THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SERIES

Volume 19 *No.* 73 *March* 1948

CONTENTS

OXFORD

AT THE CLARENDON PRESS

1948

Price 7s. 6d. net

PRINTED IN GREAT BRITAIN BY CHARLES BATEY AT THE OXFORD UNIVERSITY PRESS

THE QUARTERLY JOURNAL
OF
MATHEMATICS

OXFORD SERIES

EDITED BY

T. W. CHAUNDY U. S. HASLAM-JONES
J. H. C. THOMPSON

VOLUME XIX
1948

OXFORD
AT THE CLARENDON PRESS

Oxford University Press, Amen House, London E.C. 4
GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
BOMBAY CALCUTTA MADRAS CAPE TOWN
Geoffrey Cumberlege, Publisher to the University

PRINTED IN GREAT BRITAIN AT THE UNIVERSITY PRESS, OXFORD
BY CHARLES BATEY, PRINTER TO THE UNIVERSITY

ON THE NUMBER OF LATTICE POINTS INSIDE A RANDOM OVAL

By DAVID G. KENDALL (*Oxford*)

[Received 28 February 1947]

1. Introduction

LET $A(x)$ be the number of lattice points (μ, ν) inside or on the circle $u^2+v^2 = x$, so that $A(x) \sim \pi x$ as x tends to infinity. Let

$$R(x) = A(x) - \pi x,$$

and let ϑ be the lower bound of the numbers θ such that

$$R(x) = O(x^{\theta}).$$

It was known to Gauss† that $\vartheta \leqslant \frac{1}{2}$, and a series of investigations by W. Sierpiński,† J. G. van der Corput,† E. C. Titchmarsh (**8**), and Loo-Keng Hua (**7**) has gradually improved this inequality to $\vartheta \leqslant \frac{13}{40}$.

In the other direction, it has long been known† that $\vartheta \geqslant 0$, and this result also has been improved by G. H. Hardy (**2**), E. Landau (**6**), and A. E. Ingham (**5**) to

$$\varliminf_{x\to\infty} \frac{R(x)}{x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}} < 0, \quad \text{and} \quad \varlimsup_{x\to\infty} \frac{R(x)}{x^{\frac{1}{4}}} = +\infty.$$

Thus, in particular, employing a now well-known notation first introduced by Hardy and J. E. Littlewood, it follows that

$$R(x) = \Omega(x^{\frac{1}{4}}).$$

Lastly, there is available a series of results on the 'average order' of $R(x)$. The first of these was given by Hardy (**3**), who showed that for every $\epsilon > 0$

$$\frac{1}{x}\int_1^x |R(t)|\, dt = O(x^{\frac{1}{4}+\epsilon}), \quad \text{and} \quad \frac{1}{x}\int_1^x \{R(t)\}^2\, dt = O(x^{\frac{1}{2}+\epsilon}),$$

on the strength of which he observed: 'it is not unlikely that

$$R(x) = O(x^{\frac{1}{4}+\epsilon})$$

for all positive values of ϵ, though this has never been proved'. Hardy's 'average order' result has been improved successively by

† See the historical accounts in Landau (**6**) and Wilton (**9**).

H. Cramér (**1**) and by Landau (**6**), and it is now known, for example, that

$$\frac{1}{x}\int_1^x \{R(t)\}^2\,dt = \gamma x^{\frac{1}{2}} + O(x^{\epsilon})$$

for all positive ϵ, where γ is a known constant. But the conjecture advanced by Hardy in 1916 remains a matter for speculation.

My purpose in this paper is to discuss a number of problems related to the above which admit of a very simple and fairly complete treatment, and in which $O(x^{\frac{1}{4}})$ appears at once as the 'natural' order of the error term. I commence by discussing the lattice-point problem for a circle, with a centre no longer necessarily located at one of the points of the lattice, and prove a number of theorems to be true for 'almost all' locations of the centre of the circle. These were suggested to me by an analogy with a development in the theory of probability due to E. Borel and F. P. Cantelli, and in fact the results I shall prove can be described somewhat more colourfully in terms of the number of lattice points 'caught' inside a circular hoop thrown at random on to a chequered floor.

The methods here developed lead in a natural way to an 'explicit formula' for $A(x)$; this is related to the celebrated identity of Hardy,† which can be established as a very simple consequence, subject however to the restriction that the infinite series involved is only shown to be *summable* (j_ν), for all $\nu > \frac{1}{2}$, to the sum

$$\tfrac{1}{2}\{A(x+0)+A(x-0)\}.$$

A series $\sum a_n$ is here said to be summable (j_ν) to the sum S if

$$\lim_{\delta\to 0}\sum_{n=1}^{\infty} j_\nu(n\delta)a_n = S, \quad \text{where} \quad j_\nu(u) \equiv \frac{\Gamma(1+\nu)}{\pi^\nu}\,\frac{J_\nu(2\pi\sqrt{u})}{u^{\frac{1}{2}\nu}}.$$

In § 6 I shall prove the necessary consistency theorem for this method of summability, which is intimately related to the Bessel summability introduced by Minakshisundaram and Chandra Sekharan (**14**), in which the arguments of the Bessel functions are proportional to the integers (instead of to their square roots). They have considered the relationship between their type of summability and that of Cesàro. The analogous results for summability (j_ν), which

† Formula (3) of § 2. The first proof was given by Hardy (**2**), the identity having been stated by G. Voronoï in 1905.

do not appear to be available at present, would evidently be of interest in the lattice-point application. From another point of view, summability (j_ν) may be approached through S. Bochner's 'summability by spherical means' (**10**), devised by him for the discussion of multiple Fourier series.

Finally I shall extend the results of the first part of the paper to the more general problem in which the circle is replaced by *any* sufficiently 'smooth' oval curve, free from singularities and points of zero curvature. Similar extensions of the classical lattice-point theory have of course been given by van der Corput and V. Jarník,† but in the present problem a much simpler treatment than theirs is possible, and (as for the circle) error terms of the order $O(x^{\frac{1}{4}})$ appear as the natural outcome of the analysis.

These last results lead to a conclusion which may be of some practical importance; essentially this states that, if an oval hoop (convex, and free from singularities and points of zero curvature) is thrown at random on to the lattice plane, then the number of lattice points 'caught' will be a random variable whose mean value is equal to the area of the hoop (the unit of length being a lattice step) and whose standard deviation is effectively‡

$$a\sqrt{\left(\frac{L}{2\pi}\right)}$$

(where L is the perimeter of the hoop and $a = 0{\cdot}676497$), if the mesh of the lattice is sufficiently fine.

This makes possible a quantitative assessment of the accuracy of graphical integration by 'counting squares', when the area to be measured is bounded by an oval of the type considered. (Areas bounded in part by straight lines are thus excluded.) For example, if π were to be determined by counting the lattice points inside a circle, it follows that an accuracy of one part in a thousand would be expected if the circle contained about ten thousand lattice points. Here the circle is to be described about a centre taken at random with respect to the lattice. It will be noticed that it would be preferable to use one large circle rather than two small ones each of

† See, for example, Landau (**6**).

‡ This is for an oval hoop *possessing a centre of symmetry*. For an *asymmetric* hoop the above expression for the standard deviation should be increased by a factor $\sqrt{2}$, but then the formula gives no more than an (asymptotic) upper bound to the value of the standard deviation.

half the area, for the percentage error expected varies inversely as $A^{\frac{3}{4}}$, where A is the area of the circle.

There are immediate extensions of the results of this paper to the corresponding problems in k dimensions, but I do not propose to discuss these in detail.

2. The lattice-point problem for a random circle

Let the random variable N be the number of lattice points inside or on the boundary of a circle of radius $\sqrt{x}$ whose centre (α, β) is distributed uniformly within a single lattice cell, so that

$$N \equiv A(x;\, \alpha, \beta)$$

is the number of integer solutions of

$$(\mu-\alpha)^2+(\nu-\beta)^2 \leqslant x.$$

I shall prove that N has the mean value πx and a standard deviation σ satisfying both

$$\sigma = O(x^{\frac{1}{4}}) \quad \text{and} \quad \sigma = \Omega(x^{\frac{1}{4}}), \tag{1}$$

as x tends to infinity.

The proof makes use of the expansion of the doubly-periodic function $A(x;\, \alpha, \beta)$ in a double Fourier series,

$$A(x;\, \alpha, \beta) \sim \sum \sum a_{m,n}\, e^{2\pi i(m\alpha+n\beta)},$$

where the $(a_{m,n})$ depend on x. Let $C(u, v)$ be equal to unity or zero according as (u, v) does or does not satisfy $u^2+v^2 \leqslant x$. Then

$$A(x;\, \alpha, \beta) = \sum_{(\mu,\nu)} C(\mu-\alpha, \nu-\beta),$$

the summation being extended over all lattice points (μ, ν), although only a finite number of these contribute non-zero terms. It will follow that

$$\begin{aligned} a_{m,n} &= \int_0^1\!\!\int_0^1 A(x;\, \alpha, \beta) e^{-2\pi i(m\alpha+n\beta)}\, d\alpha d\beta \\ &= \sum_{(\mu,\nu)} \int_{\mu-1}^{\mu} \int_{\nu-1}^{\nu} C(u, v) e^{2\pi i(mu+nv)}\, dudv \\ &= \iint_{u^2+v^2 \leqslant x} \cos(2\pi mu)\cos(2\pi nv)\, dudv \\ &= \sqrt{x}\, \frac{J_1[2\pi\sqrt{x}\sqrt{(m^2+n^2)}]}{\sqrt{(m^2+n^2)}} \quad (m^2+n^2 > 0), \end{aligned} \tag{2}$$

and $a_{0,0} = \pi x$. The last step in this argument, the evaluation of what might be called 'the Fourier transform of a circle', is covered by Satz 502 of Landau's book (**6**).

Now the function $A(x;\, \alpha, \beta)$ has a complicated series of lines of discontinuity (all of them being arcs of circles of radius $\sqrt{x}$) in the (α, β)-plane, and a discussion of the convergence of its double Fourier series may therefore be expected to present some difficulty.† However, as Hobson points out, the analogue of the Riesz-Fischer theorem holds without modification, and fortunately nothing more than this is required here. Indeed, the only point requiring special attention is the proof that the double trigonometric sequence is complete, and this is an immediate consequence of Fubini's theorem and the completeness of the single trigonometric sequence.

It will be noted, in passing, that *formally*, when $\alpha = \beta = 0$, the Fourier expansion‡ gives

$$A(x) = A(x;\, 0, 0) \sim \pi x + \sqrt{x} \sum \sum{}' \frac{J_1[2\pi\sqrt{x}\sqrt{(m^2+n^2)}]}{\sqrt{(m^2+n^2)}}$$

$$\sim \pi x + \sqrt{x} \sum_{l=1}^{\infty} \frac{r(l)}{\sqrt{l}} J_1[2\pi\sqrt{(xl)}], \tag{3}$$

where $r(l)$ is the number of representations of the integer l as the sum of two squares. That this is in fact true (with a slight modification when x is an integer) is a famous result first stated by Voronoï and first proved by Hardy (**2**). I shall have more to say about this 'explicit formula' for $A(x)$ (and another, related to it) in § 5.

The Parseval identity now gives

$$\int_0^1\int_0^1 \{A(x;\, \alpha, \beta)\}^2\, d\alpha d\beta = \pi^2 x^2 + x \sum \sum{}' \frac{J_1^2[2\pi\sqrt{x}\sqrt{(m^2+n^2)}]}{m^2+n^2}$$

$$= \pi^2 x^2 + x \sum_{l=1}^{\infty} \frac{r(l)}{l} J_1^2[2\pi\sqrt{(xl)}]. \tag{4}$$

The double series is absolutely convergent because

$$|J_1(z)| < cz^{-\frac{1}{2}} \quad (z > 0),$$

† See, for example, Hardy (**11**), Titchmarsh (**13**), and E. W. Hobson (**12**).

‡ Both here and subsequently, $\sum\sum'$ means summation over all pairs (m, n) except $(0, 0)$.

and on using this inequality for the Bessel function it follows that

$$\text{Mean value } (N) \equiv \int_0^1\int_0^1 A(x;\, \alpha,\beta)\, d\alpha d\beta = a_{0,0} = \pi x,$$

and

$$\text{Variance } (N) \equiv \sigma^2 \equiv \int_0^1\int_0^1 \{A(x;\, \alpha,\beta)-\pi x\}^2\, d\alpha d\beta = O(x^{\frac{1}{2}}).$$

Thus $\sigma = O(x^{\frac{1}{4}})$, and, because

$$\sigma^2 > 4xJ_1^2(2\pi\sqrt{x}),$$

it also follows that $\sigma = \Omega(x^{\frac{1}{4}})$, as asserted in (1). This appears to be the simplest argument yet constructed which leads directly to $O(x^{\frac{1}{4}})$ as the 'true' order of the remainder-term in a lattice-point problem.

It is of interest to examine in more detail the behaviour of $\sigma/x^{\frac{1}{4}}$ for large x. In virtue of the asymptotic expansion for the Bessel function, one can write

$$\sigma^2 = \frac{x^{\frac{1}{2}}}{\pi^2}\sum_{l=1}^{\infty}\frac{r(l)}{l^{\frac{3}{2}}}\cos^2[2\pi\sqrt{(xl)}-\tfrac{3}{4}\pi]+O(1), \tag{5}$$

and so the upper and lower limits of indetermination of $\sigma^2/x^{\frac{1}{2}}$ are equal to those of

$$\frac{1}{2\pi^2}\sum_{l=1}^{\infty}\frac{r(l)}{l^{\frac{3}{2}}}\{1-\sin[4\pi\sqrt{(xl)}]\},$$

as x tends to infinity. The latter evidently lie between 0 and $2a^2$, where a is an absolute constant defined by

$$a^2 = \frac{1}{2\pi^2}\sum_{l=1}^{\infty}\frac{r(l)}{l^{\frac{3}{2}}} = \frac{2}{\pi^2}\zeta(\tfrac{3}{2})L(\tfrac{3}{2}),$$

but to improve on this statement would require a discussion of the extent to which the square roots of the integers are linearly independent.† However, the 'mean-value' result

$$\lim_{X\to\infty}\frac{1}{X}\int_1^X \frac{\sigma^2(x)}{x^{\frac{1}{2}}}\, dx = a^2, \tag{6}$$

which follows readily from the above formulae, suggests that the approximate relation

$$\sigma \doteqdot ax^{\frac{1}{4}}$$

† In this connexion see Ingham (5), 137.

can be used† to give a fair indication of the numerical average order of σ. Some practical consequences of this will be discussed in § 9. It will be shown in the Appendix that the constant a has the numerical value 0·676497.

It is clear that similar results can be established concerning the number of lattice points inside or on the boundary of a randomly located k-dimensional ellipsoid in hyperspace. For example, let E denote the ellipsoid

$$u_1^2/a_1^2+u_2^2/a_2^2+...+u_k^2/a_k^2 \leqslant x,$$

whose axes are parallel to the axes of the lattice. The 'Fourier transform' of E is

$$\iint\limits_{(E)}...\int \cos(2\pi m_1 u_1)\cos(2\pi m_2 u_2)...\cos(2\pi m_k u_k)\, du_1 du_2...du_k$$

$$= a_1 a_2...a_k \frac{x^{k/4}}{(\sum m_r^2 a_r^2)^{k/4}} J_{\frac{1}{2}k}[2\pi\sqrt{x}\sqrt{(\sum m_r^2 a_r^2)}]. \quad (7)$$

This may be proved by writing the k-fold integral in the form

$$C\iint...\int \exp(2\pi i\omega \sum b_r t_r)\, dt_1 dt_2...dt_k,$$

where $\omega^2 = \sum m_r^2 a_r^2$, $b_r = m_r a_r/\omega$, and $C = a_1 a_2...a_k$, the domain of integration being now $\sum t_r^2 \leqslant x$. An orthogonal change of variables reduces this expression to the form

$$C\iint\limits_{\sum \tau_r^2 \leqslant x}...\int \exp(2\pi i\omega\tau_1)\, d\tau_1 d\tau_2...d\tau_k,$$

which has the value

$$Cx^{k/2}\frac{\pi^{\frac{1}{2}(k-1)}}{\Gamma(\frac{1}{2}k+\frac{1}{2})}\int\limits_{-1}^{1}(1-T^2)^{\frac{1}{2}(k-1)}\exp(2\pi i\omega T\sqrt{x})\, dT,$$

in virtue of the formula for the volume of a $(k-1)$-dimensional hypersphere. The result (7) now follows on expressing the integral as a Bessel function. When all the (m_r) are zero, (7) is of course to be replaced by the volume of the ellipsoid,

$$(\pi x)^{\frac{1}{2}k} a_1 a_2...a_k/\Gamma(\tfrac{1}{2}k+1).$$

Now $|J_\nu(z)| < c_\nu z^{-\frac{1}{2}}$ for all positive z and ν. Thus, for the k-dimensional ellipsoid, with a randomly located centre and with

† It will also be noticed that the fluctuations in σ (as a function of x) could be allowed for by incorporating a 'safety factor' of amount $\sqrt{2}$.

axes parallel to those of the lattice, the mean value of the number N of included lattice points is

$$(\pi x)^{\frac{1}{2}k} a_1 a_2 \ldots a_k / \Gamma(\tfrac{1}{2}k+1),$$

and the standard deviation σ of the random variable N satisfies both

$$\sigma = O(x^{(k-1)/4}) \quad \text{and} \quad \sigma = \Omega(x^{(k-1)/4}). \tag{8}$$

Just as for the circle problem, a more accurate statement about the behaviour of σ for large x could be made, if desired, in terms of a constant analogous to the constant a above.

3. The analogue of the theorem of Borel and Cantelli

The phrasing of the results of § 2 in the language of the theory of probability suggested that one might profitably adapt an argument† employed by Borel and Cantelli in their discussion of infinite sequences of random variables.

First, it is necessary to obtain the analogue of the Bienaymé-Tchebycheff inequality. It was shown in the last section that

$$\sigma^2 \equiv \int_0^1 \int_0^1 \{R(x;\, \alpha, \beta)\}^2 \, d\alpha d\beta = O(x^{\frac{1}{2}}),$$

where $R(x;\, \alpha,\beta) = A(x;\, \alpha,\beta) - \pi x$. Thus, if, for fixed x, E is the sub-set of the unit square in which

$$|R(x;\, \alpha, \beta)| \geqslant \lambda\sigma,$$

it will follow that

$$\sigma^2 \geqslant \iint_{(E)} R^2 \, d\alpha d\beta \geqslant \lambda^2 \sigma^2 |E|,$$

and so

$$|E| \leqslant 1/\lambda^2$$

for all $\lambda > 0$, $|E|$ denoting the measure of E. The following result will now be established.

THEOREM. *Let $\lambda(x)$ be a positive function which increases steadily and without limit as x tends to infinity, and let the sequence $\{x_\nu\}$ increase so rapidly that*

$$\sum 1/\lambda_\nu^2 < \infty, \quad \textit{where} \quad \lambda_\nu \equiv \lambda(x_\nu).$$

(Such a sequence can always be found.) Then, for almost all (α, β),

$$R(x;\, \alpha, \beta) = O\{x^{\frac{1}{4}} \lambda(x)\}$$

when x tends to infinity through the sequence $\{x_\nu\}$.

† See, for example, Fréchet (**15**).

The last statement in the enunciation of the theorem is to be understood to imply, for almost all (α,β), the existence of a positive constant $k(\alpha,\beta)$ such that

$$|R(x;\,\alpha,\beta)| < k(\alpha,\beta)x^{\frac{1}{4}}\lambda(x) \quad \text{when} \quad x = x_\nu \to \infty.$$

Proof of the Theorem. Let the set V, a sub-set of the unit square, consist of the points (α,β) such that

$$|R(x_\nu;\,\alpha,\beta)| < \lambda_\nu \sigma_\nu$$

for all sufficiently large ν, where $\sigma_\nu \equiv \sigma(x_\nu) = O(x_\nu^{\frac{1}{4}})$. Then the theorem asserts that $|V| = 1$ if $\sum 1/\lambda_\nu^2$ is convergent. Now, an asterisk here denoting complementation with regard to the unit square,

$$V = \sum_{\nu=1}^{\infty} S_\nu,$$

where $$S_\nu = E_\nu^* E_{\nu+1}^* E_{\nu+2}^* \ldots,$$

and $$E_\nu \equiv \text{Set}\,\{|R(x_\nu;\,\alpha,\beta)| \geqslant \lambda_\nu \sigma_\nu\},$$

and so $$|V| \geqslant |S_\nu| = 1-|S_\nu^*|, \quad \text{for all } \nu.$$

But $$S_\nu^* = E_\nu + E_{\nu+1} + E_{\nu+2} + \ldots,$$

and so

$$|S_\nu^*| \leqslant \sum_{\mu=\nu}^{\infty} |E_\mu| \leqslant \sum_{\mu=\nu}^{\infty} 1/\lambda_\mu^2 < \delta, \quad \text{if} \quad \nu \geqslant \nu_0(\delta), \quad \text{for any } \delta > 0.$$

Thus $|V| > 1-\delta$ for all $\delta > 0$, and the theorem is proved.

Particular cases.

(i) $\lambda(x) = x^\epsilon$ $(\epsilon > 0)$. Then, if the sequence $\{x_\nu\}$ has zero as its exponent of convergence, i.e. if $\sum x_\nu^{-\rho}$ is convergent for all $\rho > 0$, the statement

$$R(x;\,\alpha,\beta) = O(x^{\frac{1}{4}+\epsilon})$$

will be true, for almost all (α,β), as x tends to infinity through the sequence $\{x_\nu\}$.

This result can be strengthened. For let H_r be the exceptional (α,β)-set (of measure zero) when $\epsilon = 1/r$. Then $\sum H_r$ will also be of measure zero, and so for almost all (α,β)

$$R(x;\,\alpha,\beta) = O(x^{\frac{1}{4}+\eta}) \quad \textit{for all} \quad \eta > 0, \tag{9}$$

as x tends to infinity through the sequence $\{x_\nu\}$. Here the implied constant will in general depend on α, β, and η.

(ii) The order of the remainder-term can be still further reduced

if the rate of increase of the sequence $\{x_\nu\}$ is accelerated sufficiently. For example, if

$$x_\nu = 2^{2^\nu},$$

then for almost all (α, β),

$$R(x;\alpha,\beta) = O\{x^{\frac{1}{4}}(\log x)^{\epsilon}\} \quad \textit{for all} \quad \epsilon > 0, \tag{10}$$

as x tends to infinity through the sequence $\{x_\nu\}$. It is of interest to compare this with Hardy's theorem that

$$\varliminf_{x\to\infty} \frac{R(x)}{x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}} < 0, \tag{11}$$

where $R(x) = R(x; 0, 0)$.

Evidently, while Hardy's result (11) *may* also be true for $R(x;\alpha,\beta)$ when α and β are not both zero, it *cannot* be true that there exists a fixed sequence $\{x_n\}$ such that

$$R(x;\alpha,\beta) < -K(\alpha,\beta)x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}, \qquad K(\alpha,\beta) > 0,$$

as x tends to infinity through the sequence $\{x_n\}$, for all (α,β) in a set of positive measure. For, if this were so, the sequence $\{x_n\}$ would contain a sub-sequence $\{x_\nu\}$ with a rate of growth sufficiently rapid to ensure that

$$R(x;\alpha,\beta) = o\{x^{\frac{1}{4}}(\log x)^{\frac{1}{4}}\} \quad (x = x_\nu \to \infty)$$

for almost all (α,β), and this would give a contradiction.

The relationship between (10) and (11) may be viewed in another way. From (11), there exists a sequence $\{x_n\}$ such that

$$R(x) < -Kx^{\frac{1}{4}}(\log x)^{\frac{1}{4}} \quad (K > 0;\ x = x_n \to \infty).$$

This contains a sub-sequence $\{x_\nu\}$, which can be chosen in an infinity of ways, such that

$$|R(x;\alpha,\beta)| < k(\alpha,\beta)x^{\frac{1}{4}}(\log x)^{\frac{1}{5}} \quad (x = x_\nu \to \infty),$$

unless (α,β) lies in a set H (of measure zero) which depends on the choice of the sub-sequence. It then follows that the origin $(0, 0)$ must be a member of all such sets H.

4. Lattice 'points' of finite size

There is another analogue of the classical lattice-point problem which can be solved by the methods of this paper. Let the lattice points (μ,ν) be replaced by small equal circular spots, say

$$(u-\mu)^2+(v-\nu)^2 \leqslant \delta,$$

where δ is small and positive. Instead of the function $A(x;\alpha,\beta)$ one

now considers $\pi\delta B_\delta(x;\alpha,\beta)$, the total area of the lattice spots included within the original circle

$$(u-\alpha)^2+(v-\beta)^2 \leqslant x,$$

only the area actually included being counted for spots which overlap the boundary. Then

$$\pi\delta B_\delta(x;\alpha,\beta) = \sum_{(\mu,\nu)} \iint_{u^2+v^2\leqslant\delta} C(\mu+u-\alpha,\nu+v-\beta)\,dudv$$

$$= \iint_{u^2+v^2\leqslant\delta} A(x;\alpha-u,\beta-v)\,dudv,$$

and in particular

$$\pi\delta B_\delta(x) \equiv \pi\delta B_\delta(x;0,0) = \iint_{u^2+v^2\leqslant\delta} A(x;u,v)\,dudv$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} A(\delta;u,v)A(x;u,v)\,dudv$$

$$= \pi^2\delta x+\surd(\delta x)\sum\sum{}' \frac{J_1[2\pi\surd x\surd(m^2+n^2)]J_1[2\pi\surd\delta\surd(m^2+n^2)]}{m^2+n^2}, \tag{12}$$

by the Parseval identity, the double series being absolutely convergent. Thus

$$B_\delta(x)-\pi x = O(x^{\frac{1}{4}}) \tag{13}$$

as x tends to infinity. This is true for all fixed $\delta > 0$, but not of course uniformly with regard to δ, so that one cannot proceed to the parallel statement (known to be false) for the lattice-*point* problem by allowing δ to tend to zero.

5. An 'explicit formula' for $A(x)$, and its relation to Hardy's identity

The double series (12) is absolutely convergent, and so

$$B_\delta(x) = \frac{1}{\pi\delta}\iint_{u^2+v^2\leqslant\delta} A(x;u,v)\,dudv = \pi x+\surd x\sum_{l=1}^{\infty} j_1(l\delta)\frac{r(l)}{\surd l}J_1[2\pi\surd(xl)],$$

where

$$j_1(u) \equiv \frac{J_1(2\pi\surd u)}{\pi\surd u}.$$

Now, if x is not an integer, $A(x;u,v)$ will be constant and equal to $A(x)$ for all (u,v) within the circle $u^2+v^2 \leqslant \delta$, provided that δ is

small enough. This will be true, for example, if $0 < \delta < \delta_0(x)$, where $\delta_0(x)$ is the smaller of

$$\{\sqrt{x}-\sqrt{[x]}\}^2 \quad \text{and} \quad \{\sqrt{([x]+1)}-\sqrt{x}\}^2.$$

When $x-[x]$ is equal to a constant positive proper fraction, $\delta_0(x) = O(1/x)$ as x tends to infinity. Thus

$$A(x) = \pi x+\sqrt{x}\sum_{l=1}^{\infty} j_1(l\delta)\frac{r(l)}{\sqrt{l}} J_1[2\pi\sqrt{(xl)}], \tag{14}$$

if x is not an integer, for *all* δ satisfying $0 < \delta < \delta_0(x)$. This provides an 'explicit formula' for $A(x)$ as an absolutely convergent series of Bessel functions. It is clearly related to Hardy's identity (3), and a weak form of the latter can be deduced from it in the following way.

I shall say that a series $\sum a_n$ is 'summable (j_ν)' to the sum S if

$$\lim_{\delta\to 0}\sum_{n=1}^{\infty} j_\nu(n\delta)a_n = S,$$

where
$$j_\nu(u) = \frac{\Gamma(1+\nu)}{\pi^\nu}\frac{J_\nu(2\pi\sqrt{u})}{u^{\frac{1}{2}\nu}}. \tag{15}$$

It will be shown in §6 that summability (j_ν) is *regular* for $\nu > \frac{1}{2}$, so that, if $\sum a_n$ is convergent in the ordinary sense to the sum S, then it is summable (j_ν) to S. Thus from (14) it can at once be concluded that (3) is true *if* the series is known to be convergent (x being still restricted to non-integer values). Hardy and Landau (4) have already pointed out that the identity (3) can be obtained fairly easily if only summability (A) is required, and that the real difficulty lies in establishing the convergence of the series. The above method, perhaps a little shorter than that of Hardy and Landau because no use is made of the singular integral of Weierstrass, leads to a somewhat stronger partial result, the convergence factor introduced being only $O(n^{-\frac{3}{4}})$, instead of $O(e^{-n\delta})$ as in Abel summability.

It is also worth noting that the result can be strengthened by a slight variation in the procedure. One considers

$$\iint\limits_{u^2+v^2\leqslant\delta} A(x;u,v)H(u,v)\,dudv,$$

where $H(u,v) = (\delta-u^2-v^2)^{-\frac{1}{2}+\epsilon}$ inside the circle $u^2+v^2 = \delta$ and is zero elsewhere, and $\epsilon > 0$; $H(u,v)$ then belongs to the class L^2, and

the Parseval identity can be applied as before. The Fourier coefficients of the function $H(u,v)$ are†

$$\begin{aligned} b_{m,n} &= \iint\limits_{u^2+v^2\leqslant\delta} H(u,v)e^{2\pi i(mu+nv)}\,dudv \\ &= \iint\limits_{\rho^2\leqslant\delta} (\delta-\rho^2)^{-\frac{1}{2}+\epsilon}e^{2\pi i\sqrt{(m^2+n^2)}\rho\cos\phi}\,\rho d\rho d\phi \\ &= 2\pi\delta^{\frac{1}{2}+\epsilon}\int\limits_0^1 (1-t^2)^{-\frac{1}{2}+\epsilon}J_0[2\pi t\sqrt{\delta}\sqrt{(m^2+n^2)}]\,tdt \\ &= \frac{G(\epsilon,\delta)\Gamma(\frac{3}{2}+\epsilon)}{\pi^{\frac{1}{2}+\epsilon}\delta^{\frac{1}{4}+\frac{1}{2}\epsilon}}\,\frac{J_{\frac{1}{2}+\epsilon}[2\pi\sqrt{\delta}\sqrt{(m^2+n^2)}]}{(m^2+n^2)^{\frac{1}{4}+\frac{1}{2}\epsilon}} \qquad (m^2+n^2>0), \end{aligned}$$

where
$$G(\epsilon,\delta) = \iint\limits_{u^2+v^2\leqslant\delta} H(u,v)\,dudv = b_{0,0}.$$

As before, if x is not an integer, it follows that

$$A(x) = \pi x+\sqrt{x}\sum_{l=1}^{\infty} j_{\frac{1}{2}+\epsilon}(l\delta)\frac{r(l)}{\sqrt{l}}J_1[2\pi\sqrt{(xl)}], \tag{16}$$

for all $\epsilon>0$, and for *all* sufficiently small δ. Thus the right-hand side of (3) is summable (j_ν) to $A(x)$ for all $\nu>\frac{1}{2}$.

By varying the form of $H(u,v)$ a variety of such results can be established. Similar methods were considered by S. Bochner (**10**) in connexion with the summation of multiple Fourier series, but he was more interested in choosing functions $H(u,v)$ which would lead to familiar methods of summability, and he does not discuss summability (j_ν).

Even if x is an integer, the explicit formulae (14) and (16) will still be true if x is not expressible as the sum of two squares. In fact, the precise condition is that the open interval bounded by

$$(\sqrt{x}\pm\sqrt{\delta})^2 = x+\delta\pm 2\sqrt{(x\delta)}$$

should contain no integer values $x = M$ such that $r(M)>0$.

When $x = M$, where $r(M)>0$, there is nothing to correspond to the explicit formulae (14) and (16), but it may readily be seen that the summability results are still valid if the appropriate modification is made to the definition of $A(x)$. Corresponding to each pair of lattice points (μ,ν) and $(-\mu,-\nu)$ on the perimeter of the circle

† See, for example, G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1944), 48 and 373.

$u^2+v^2 = x$ there are two circular arcs-of-discontinuity for the function $A(x; u, v)$ which touch externally at the origin, each being of radius $\sqrt{x}$. They divide the interior of the smaller circle $u^2+v^2 = \delta$, for sufficiently small δ, into a finite number of 'cuspidal' regions of area at most $O(\delta^{\frac{3}{2}})$, and a finite number of convex regions in which

$$A(x; u, v) = \tfrac{1}{2}\{A(x+0)+A(x-0)\}.$$

Thus

$$\lim_{\delta\to 0} \frac{1}{\pi\delta} \iint\limits_{u^2+v^2\leqslant\delta} A(x; u, v)\, dudv = \tfrac{1}{2}\{A(x+0)+A(x-0)\},$$

and so the summability results are true for general values of x if in the definition of $A(x)$ all the lattice points which fall on the boundary are counted with a weight $\frac{1}{2}$.

Preliminary examination of the explicit formula (16) suggests that it is *not* likely to be very productive of results concerning the classical lattice-point problem. To illustrate this, I shall employ it to prove: *if $r(M) > 0$, and if M' is the next largest integer for which $r(M') > 0$, then $M'-M = O(M^{\frac{3}{8}+\epsilon})$, for all $\epsilon > 0$.*

Let $\Delta = M'-M$, $x_1 = M+\frac{1}{3}\Delta$, and $x_2 = M+\frac{2}{3}\Delta$. From (16), on replacing the Bessel functions by the first terms in their asymptotic expansions, it follows that

$$|R(x)| \equiv |A(x)-\pi x| < K_\epsilon x^{\frac{1}{4}}/\delta^{\frac{1}{2}+\frac{1}{2}\epsilon},$$

where K_ϵ is a positive constant. This inequality is to be used for $x = x_1$ and x_2, and so for δ it is permissible to take any value (and preferably the largest) such that both the open intervals $(\sqrt{x_1}\pm\sqrt{\delta})^2$ and $(\sqrt{x_2}\pm\sqrt{\delta})^2$ exclude both the points $x = M$ and $x = M'$. The choice

$$\delta = \frac{\Delta^2}{36(M+\Delta)}$$

may easily be seen to satisfy these conditions, and so

$$\tfrac{1}{3}\pi\Delta = |R(x_2)-R(x_1)| < \frac{K_\epsilon 6^{1+\epsilon}(M+\Delta)^{\frac{1}{2}+\frac{1}{2}\epsilon}}{\Delta^{1+\epsilon}}\{(M+\tfrac{1}{3}\Delta)^{\frac{1}{4}}+(M+\tfrac{2}{3}\Delta)^{\frac{1}{4}}\}.$$

Now, if $M = a^2+b^2$, where $a \geqslant 0$, then

$$(a+1)^2+b^2 = 2M-\{(a-1)^2+b^2-2\} \leqslant 2M$$

if $M > 4$, and so certainly $M'-M = \Delta = O(M)$. Thus

$$\Delta^{2+\epsilon} = O(M^{\frac{3}{4}+\frac{1}{2}\epsilon}) \quad \text{for all } \epsilon > 0,$$

and this is equivalent to the result stated.

This is of course a fairly crude result, for the combination of $R(x) = O(x^{\theta})$, $\Delta = O(M)$, and $\frac{1}{3}\pi\Delta = |R(x_2)-R(x_1)|$ gives at once

$$M'-M = O(M^{\theta}).$$

Much closer estimates of the order of $M'-M$ can thus be obtained by means of any of the modern results on the order of $R(x)$, from Sierpiński's $\vartheta \leqslant \frac{1}{3}$ to Loo-Keng Hua's $\vartheta \leqslant \frac{13}{40}$. The alternative application of (16) to give an estimate of the order of $R(x)$ seems to be even less promising, for it would be necessary to place a lower bound to Δ, the difference between consecutive members of the sequence of integers expressible as the sum of two squares.

It will be noted that the infinite series in (16) (for $\epsilon > 0$) is absolutely and uniformly convergent within any closed interval of the x-axis which excludes all integer values $x = M$ such that $r(M) > 0$. This is in accord with the continuity of the sum-function $A(x)$ except at such points $x = M$.

6. The regularity of the method of summability (j_ν) for $\nu > \frac{1}{2}$

From the definition of $j_\nu(u)$ it is clear that

$$\lim_{u\to 0} j_\nu(u) = 1 \quad \text{and} \quad \lim_{u\to\infty} j_\nu(u) = 0$$

for $\nu > -\frac{1}{2}$. Let

$$c_n(\delta) \equiv j_\nu(n\delta)-j_\nu\{(n+1)\delta\};$$

then from a theorem of O. Toeplitz† it follows that the (j_ν) method will be *regular* if and only if

(i) $\sum |c_n(\delta)| \leqslant K$ for all $\delta > 0$;

(ii) $c_n(\delta) \to 0$ as $\delta \to 0$, for each n;

(iii) $\lim_{\delta\to 0} \sum_{n=1}^{\infty} c_n(\delta) = 1.$

The verification of the second and third conditions is immediate.

The first is satisfied if $\nu > \frac{1}{2}$ because then

$$c_n(\delta) = -\frac{\Gamma(1+\nu)}{\pi^\nu} \int\limits_{n\delta}^{(n+1)\delta} \frac{d}{du}\left\{\frac{J_\nu(2\pi\sqrt{u})}{u^{\frac{1}{2}\nu}}\right\} du$$

$$= 2^\nu\Gamma(1+\nu) \int\limits_{2\pi\sqrt{(n\delta)}}^{2\pi\sqrt{\{(n+1)\delta\}}} J_{\nu+1}(t)t^{-\nu}\,dt,$$

† See, for example, D. V. Widder (**17**). The distinction between the discrete form of the theorem (proved by Widder) and the continuous form used here is not important.

and $$\int_0^\infty J_{\nu+1}(t)t^{-\nu}\,dt$$

is absolutely convergent.

Mention must here be made of a very similar type of Bessel summability introduced by Minakshisundaram and considered in further detail by K. Chandra Sekharan (**14**). This differs from summability (j_ν) in that the arguments of the Bessel functions are proportional to the integers themselves instead of to their square roots. Chandra Sekharan has given a number of results relating this summability (J_μ) to the Cesàro methods. It seems worth observing that a similar discussion of (j_ν) summability might find useful applications in lattice-point theory, particularly in establishing the 'explicit formulae' in the associated k-dimensional problems.

7. Extension of the preceding results to general oval curves

Lattice-point problems for more general oval curves than the circle have been considered by several writers, notably J. G. van der Corput and V. Jarník.† It is therefore of interest that the results of the present paper can be extended by quite a simple argument to a very wide class $\mathfrak{C}$ of oval curves, defined as follows.

(i) An oval of class $\mathfrak{C}$ in the (u,v)-plane is cut by the line

$$u\cos\psi+v\sin\psi = p$$

in just two points if $-f(\psi+\pi) < p < f(\psi)$, while there are no intersections if $p < -f(\psi+\pi)$ or if $p > f(\psi)$. For the two critical values of p, the line is a tangent to the oval. It will be assumed that $f(\psi) > 0$ for all ψ, so that the origin is an 'interior' point.

(ii) The function $f(\psi)$ has the period 2π, and possesses a bounded fourth differential coefficient. Thus $f(\psi)$ is itself bounded, with all intermediate derivatives. It will be necessary to make the further assumption that

$$f(\psi)+f''(\psi) > 0 \quad \text{for all } \psi. \tag{17}$$

From (i), the line

$$u\cos\psi+v\sin\psi = f(\psi) \tag{18}$$

is always a tangent to the oval; let the point of contact be

$$\{f(\psi)\cos\psi-g(\psi)\sin\psi;\ f(\psi)\sin\psi+g(\psi)\cos\psi\}; \tag{19}$$

† See, for example, Landau (**6**) and Wilton (**9**).

then on identifying (18) with the tangent to the locus of (19) it will be seen that $g(\psi) \equiv f'(\psi)$; i.e. the oval has the parametric equations

$$u = f(\psi)\cos\psi - f'(\psi)\sin\psi, \qquad v = f(\psi)\sin\psi + f'(\psi)\cos\psi.$$

Thus
$$du/d\psi = -\{f(\psi)+f''(\psi)\}\sin\psi,$$
and
$$dv/d\psi = \{f(\psi)+f''(\psi)\}\cos\psi,$$
and so the radius of curvature at the point with parameter ψ is

$$\rho = f(\psi)+f''(\psi). \tag{20}$$

From (17), and the continuity of $f(\psi)$ and its derivatives, it therefore follows that ρ has a positive lower bound. Also

$$f^{\mathrm{iv}}(\psi) = \frac{d^2\rho}{d\psi^2} - \rho + f(\psi), \quad \text{and} \quad \frac{d^2\rho}{ds^2} = \frac{1}{\rho^2}\frac{d^2\rho}{d\psi^2} - \frac{1}{\rho^3}\left(\frac{d\rho}{d\psi}\right)^2.$$

Thus the conditions defining an oval of class $\mathfrak{C}$ are essentially that it is to be a bounded closed convex curve, free from singular points† and from points of zero curvature (for which ρ would be infinite), while the radius of curvature must possess a bounded second differential coefficient with respect to the arc length.

In particular, the following are *not* of class $\mathfrak{C}$: the asteroid

$$u^{\frac{2}{3}} + v^{\frac{2}{3}} = 1,$$

which is non-convex and possesses four cusps; and the (convex) quartic oval

$$u^2 + v^4 = 1,$$

which has points of zero curvature at $(1, 0)$ and $(-1, 0)$.

Let $N \equiv A(x; \alpha, \beta)$ be the number of lattice points inside or on the oval

$$u = x^{\frac{1}{2}}\{f(\psi)\cos\psi - f'(\psi)\sin\psi\} + \alpha,$$
$$v = x^{\frac{1}{2}}\{f(\psi)\sin\psi + f'(\psi)\cos\psi\} + \beta,$$

which is similar to the first (the scale factor being $\sqrt{x}:1$), and which has been displaced through (α, β) in directions parallel to the principal axes of the lattice. If (α, β) is uniformly distributed within any cell of the lattice, N will be a random variable with a mean value equal to x multiplied by the area of the original 'unit' oval. I shall prove that, exactly as for the circle, the standard deviation σ of N satisfies both

$$\sigma = O(x^{\frac{1}{4}}) \quad \text{and} \quad \sigma = \Omega(x^{\frac{1}{4}}), \tag{21}$$

when x tends to infinity. I shall also obtain a more precise estimate of the order of magnitude of σ, in terms of the maximum radius of

† At a singular point $du/d\psi = dv/d\psi = 0$, and this would contradict (17).

curvature of the unit oval. The method is based on a theorem of E. C. Titchmarsh (**16**) concerning the order of magnitude of a Fourier transform, but it will be necessary to push the analysis a little farther in order to obtain an estimate for the remainder term in his formula. Before proceeding to the details, one further general point may be noted. The theorem of § 3 was an immediate deduction from the result $\sigma = O(x^{\frac{1}{4}})$ for a circle. Thus *the theorem is equally true for the general oval of class* $\mathfrak{C}$. $R(x;\alpha,\beta)$ is then the difference between the number of lattice points 'caught' in the oval, and its area (the area of the unit oval multiplied by x).

The 'Fourier transform' of the oval is the double sequence

$$a_{m,n} = x \iint\limits_{(O_1)} e^{2\pi i(mu+nv)\sqrt{x}}\, dudv,$$

where O_1 denotes the interior of the unit oval. It will be noticed that in general $a_{m,n}$ will not be a real number. If one writes

$$m^2+n^2 = l \neq 0, \quad m = l^{\frac{1}{2}}\cos\psi_0, \quad \text{and} \quad n = l^{\frac{1}{2}}\sin\psi_0,$$

then $$mu+nv = l^{\frac{1}{2}}(u\cos\psi_0+v\sin\psi_0),$$

and so, on writing $F(\phi) = f(\phi+\psi_0)$, $\phi = \psi-\psi_0$, and on taking a new set of axes inclined at an angle ψ_0 to the first, one will have

$$a_{m,n} = x \iint\limits_{(O_1)} e^{2\pi i\sqrt{(lx)}U}\, dUdV,$$

where the equations to the unit oval are now

$$U = F(\phi)\cos\phi - F'(\phi)\sin\phi, \qquad V = F(\phi)\sin\phi + F'(\phi)\cos\phi,$$

in the new coordinates (U, V). It is important to bear in mind throughout the following analysis that all the initial assumptions concerning $f(\psi)$ are also true of $F(\phi)$, *uniformly with regard to* ψ_0.

It now follows that

$$\begin{aligned} a_{m,n} &= -x \oint e^{2\pi i\sqrt{(lx)}U}\, VdU \\ &= -\frac{1}{2\pi i}\sqrt{\left(\frac{x}{l}\right)} \oint e^{2\pi i\sqrt{(lx)}U} \cot\phi\, dU, \end{aligned} \tag{22}$$

on integrating by parts and writing $\cot\phi$ for $-dV/dU$, and it is a question of finding a suitable asymptotic formula for the last integral, valid for large values of lx. It will be noticed that the integral, although convergent, is improper because of the infinities of $\cot\phi$ when $\phi = 0$ or π.

It is convenient to divide the range of integration at the points $\phi = 0, \delta, \frac{1}{2}\pi, \pi-\delta', \pi, \pi+\delta'', \frac{3}{2}\pi$, and $2\pi-\delta'''$; of the corresponding contributions to the integral the following are typical:

$$J_1 = \int\limits_{\phi=0}^{\delta} e^{2\pi i\sqrt{(lx)}U} \cot\phi \, dU, \quad \text{and} \quad J_2 = \int\limits_{\phi=\delta}^{\frac{1}{2}\pi} e^{2\pi i\sqrt{(lx)}U} \cot\phi \, dU.$$

In order to obtain an inequality for J_2 it is enough to observe that

$$dU/d\phi = -\{F(\phi)+F''(\phi)\}\sin\phi,$$

which is negative when $\delta \leqslant \phi \leqslant \frac{1}{2}\pi$, in virtue of (17) (which is a consequence of the absence of singular points from the curve bounding the oval). It follows that $\cot\phi$ is a monotonic increasing function of U for values of ϕ in the interval $(\delta, \frac{1}{2}\pi)$, and so, by the second mean-value theorem,

$$J_2 = (\cot\delta)O\{1/\sqrt{(lx)}\} = O\left\{\frac{1}{\delta\sqrt{(lx)}}\right\}, \tag{23}$$

uniformly with regard to ψ_0.

The discussion of J_1 is a little more complicated. It is first necessary to show that the expression

$$\cot\phi - \sqrt{\left\{\frac{\frac{1}{2}\rho_0}{F(0)-U}\right\}} \tag{24}$$

is bounded in $0 \leqslant \phi \leqslant \Delta$, for a sufficiently small value of Δ, uniformly with regard to ψ_0. Here

$$\rho_0 = F(0)+F''(0)$$

is the radius of curvature at that point of the perimeter of the oval whose parameter was ψ_0 with respect to the original pair of axes, and so

$$\rho_0 \geqslant b \equiv \min\{f(\psi)+f''(\psi)\} > 0.$$

The expression (24) can be written in the form

$$\frac{\{F(0)-U\}-\frac{1}{2}\rho_0\tan^2\phi}{\tan\phi\sqrt{\{F(0)-U\}}[\sqrt{\{F(0)-U\}}+\sqrt{(\frac{1}{2}\rho_0)}\tan\phi]}, \tag{25}$$

and the numerator of this is $O(\phi^3)$, if Δ is less than say $\frac{1}{4}\pi$, because

$$U = F(0)-\tfrac{1}{2}\rho_0\phi^2+\tfrac{1}{6}\phi^3U'''(\lambda\phi) \quad (0 < \lambda < 1),$$

and

$$U'''(\phi) = \{F(\phi)+F''(\phi)\}\sin\phi-2\{F'(\phi)+F'''(\phi)\}\cos\phi- \\ -\{F''(\phi)+F^{\text{iv}}(\phi)\}\sin\phi,$$

which is by hypothesis bounded for all ϕ and ψ_0. On the other hand,

$$F(0)-U = \tfrac{1}{2}\rho_0\phi^2\{1+O(\phi)\},$$

since $\rho_0 \geqslant b$, and so

$$\sqrt{\{F(0)-U\}} \geqslant k\phi \quad (0 \leqslant \phi \leqslant \Delta;\ k > 0)$$

if Δ is sufficiently small, where the implied constants only involve $b = \min\rho(\psi)$ and so are independent of ψ_0. Thus the denominator of (25) is greater than or equal to some positive multiple of ϕ^3 in $0 \leqslant \phi \leqslant \Delta$, and this proves the result. It at once follows that

$$\int\limits_{\phi=0}^{\delta} e^{2\pi i\sqrt{(lx)}U}\left\{\cot\phi - \frac{\sqrt{\{\frac{1}{2}\rho_0\}}}{\sqrt{\{F(0)-U\}}}\right\} dU = O\{F(0)-U(\delta)\} = O(\delta^2), \quad \text{if } \delta \leqslant \Delta, \tag{26}$$

uniformly with regard to ψ_0.

Finally,

$$\sqrt{(\tfrac{1}{2}\rho_0)}\int\limits_{\phi=0}^{\delta} e^{2\pi i\sqrt{(lx)}U}\frac{dU}{\sqrt{\{F(0)-U\}}}$$

$$= -\frac{\rho_0^{\frac{1}{2}}}{2(lx)^{\frac{1}{4}}}e^{2\pi i\sqrt{(lx)}F(0)-i\pi/4} + O\sqrt{\left\{\frac{\frac{1}{2}\rho_0}{lx[F(0)-U(\delta)]}\right\}},$$

while by the argument just completed the remainder term in this formula is

$$O\left\{\frac{1}{\delta\sqrt{(lx)}}\right\}. \tag{27}$$

In virtue of (23), (26), (27), it has therefore been shown that

$$J_1+J_2 = -\frac{\rho_0^{\frac{1}{2}}}{2(lx)^{\frac{1}{4}}}e^{2\pi i\sqrt{(lx)}F(0)-i\pi/4} + O\left\{\frac{1}{\delta\sqrt{(lx)}}\right\} + O(\delta^2),$$

uniformly with regard to ψ_0, and here δ can be given any convenient value less than Δ. The two remainder terms will be of the same order if $\delta = (lx)^{-\frac{1}{6}}$, and this will be less than Δ if x is large enough, for all $l \geqslant 1$. Thus the combined error term is

$$O\{(lx)^{-\frac{1}{3}}\}.$$

The remaining contributions to (22) can be estimated in the same way, and the final result is

$$a_{m,n} = \frac{x^{\frac{1}{4}}}{2\pi i l^{\frac{3}{4}}}\{\rho_0^{\frac{1}{2}}e^{2\pi i\sqrt{(lx)}p_0-i\pi/4} - \rho_\pi^{\frac{1}{2}}e^{-2\pi i\sqrt{(lx)}p_\pi+i\pi/4}\} + O(x^{\frac{1}{6}}/l^{\frac{5}{6}}), \tag{28}$$

as x tends to infinity. Here $l = m^2+n^2 > 0$, the implied constant is independent of m and n (this is a consequence of the uniformity

with regard to ψ_0), $p = f(\psi)$, and the suffixes 0 and π refer to the values $\psi = \psi_0$ and $\psi = \psi_0 + \pi$, respectively, where

$$m = l^{\frac{1}{2}} \cos \psi_0 \quad \text{and} \quad n = l^{\frac{1}{2}} \sin \psi_0.$$

It is now possible, as in § 2, to investigate the behaviour of σ (the standard deviation of N) when x is large, because as before

$$\sigma^2 = \sum \sum' |a_{m,n}|^2,$$

the summation being over all pairs (m, n) except $(0, 0)$. Thus immediately it can be seen that $\sigma = O(x^{\frac{1}{4}})$, while from $\sigma > |a_{m,n}|$ it follows that $\sigma = \Omega(x^{\frac{1}{4}})$, as already asserted in (21), and so the theorem of § 3 applies also to ovals of class $\mathfrak{C}$.

It will also be seen that

$$\sigma^2 \leqslant \frac{x^{\frac{1}{2}} \rho_{\max}}{\pi^2} \sum_{l=1}^{\infty} \frac{r(l)}{l^{\frac{3}{2}}} + O(x^{\frac{5}{12}}) = 2a^2 x^{\frac{1}{2}} \rho_{\max} + O(x^{\frac{5}{12}}), \tag{29}$$

where a is the constant 0·676497 introduced in § 2, and $\rho_{\max}$ is the maximum radius of curvature of the unit oval (so that $x^{\frac{1}{2}} \rho_{\max}$ is the maximum radius of curvature of the *actual* oval).

When the oval possesses a centre of symmetry, this can be taken as origin, and the asymptotic formula then becomes

$$a_{m,n} = \frac{x^{\frac{1}{4}} \rho^{\frac{1}{2}}}{\pi l^{\frac{3}{4}}} \cos\{2\pi\sqrt{(lx)}p - \tfrac{3}{4}\pi\} + O(x^{\frac{1}{6}}/l^{\frac{5}{6}}), \tag{28 a}$$

where now ρ and p each refer to the point $\psi = \psi_0$. It is of interest to compare this with the exact value (2) for the circle (when $p = \rho = 1$). It will be found that the error term is actually much smaller than that indicated by (28 a); evidently, in proving the latter, something has had to be sacrified in order to achieve the required generality.

It follows that for a centrally symmetric oval of class $\mathfrak{C}$,

$$\sigma^2 = \frac{x^{\frac{1}{2}}}{2\pi^2} \sum \sum' \frac{\rho}{l^{\frac{3}{2}}} \{1 - \sin[4\pi\sqrt{(lx)}p]\} + O(x^{\frac{5}{12}}); \tag{30}$$

here it is not possible to replace the double summation by a single summation over the values of $l = m^2 + n^2$, because p and ρ each depend on $\psi_0 = \tan^{-1}(n/m)$. However, if the *orientation* of the oval is also allowed to vary randomly, so that it is uniformly distributed over the interval $(0, 2\pi)$, a much simpler result can be obtained.

Let $\bar{\sigma}$ be the standard deviation of N, the number of lattice points 'caught', when the varying orientation is allowed for. The mean value of N is unchanged, and so

$$\bar{\sigma}^2 = \frac{1}{2\pi}\int\limits_0^{2\pi} \sigma^2\, d\psi.$$

Here ψ is being used to denote the angle between a fixed direction in the lattice and a direction fixed with respect to the oval, and evidently

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \rho\, d\psi = \lambda/2\pi, \quad \text{for all } m \text{ and } n,$$

where λ is the perimeter of the unit oval. It will then be found from (30) that

$$\lim_{X\to\infty} \frac{1}{X}\int\limits_1^X \frac{\bar{\sigma}^2(x)}{x^{\frac{1}{2}}}\, dx = \frac{\lambda a^2}{2\pi}, \tag{31}$$

where a is the constant 0·676497. This is the generalization of equation (6) to centrally symmetric ovals of class $\mathfrak{C}$.

Returning to the general (not necessarily symmetric) oval of class $\mathfrak{C}$, it follows from (28) that

$$|a_{m,n}| \leqslant \frac{x^{\frac{1}{4}}}{2\pi l^{\frac{3}{4}}}(\rho_0^{\frac{1}{2}}+\rho_\pi^{\frac{1}{2}})+O(x^{\frac{1}{6}}/l^{\frac{5}{6}}),$$

and so, on using the Schwarz inequality, one obtains

$$\bar{\sigma}^2 \leqslant 2x^{\frac{1}{2}}\left(\frac{\lambda a^2}{2\pi}\right)+O(x^{\frac{5}{12}}). \tag{32}$$

When the general oval is being considered, therefore, the order of magnitude for the standard deviation $\bar{\sigma}$ suggested by (31) should be increased by a factor $\sqrt{2}$. The formula (32) should also be used, whatever the form of the oval, if it is desired to make allowance for the irregular fluctuations in $\bar{\sigma}$ (regarded as a function of x).

8. The behaviour of σ for ovals not of class $\mathfrak{C}$

It has already been noted that the quartic oval

$$u^2x+v^4 \leqslant x^2$$

is not of class $\mathfrak{C}$ because it possesses two points of zero curvature. In this instance the 'Fourier transform' of the oval can be examined

in detail, and it is very instructive to see how the general theory of the preceding section fails in this case. It is only necessary to evaluate the 'Fourier transform'

$$a_{m,n} = \iint\limits_{u^2x+v^4\leqslant x^3} \cos(2\pi mu)\cos(2\pi nv)\, du dv$$

when $n = 0$, and then

$$a_{m,0} = \frac{2\Gamma(\frac{5}{4})}{\pi^{\frac{1}{4}}} \frac{J_{\frac{3}{4}}(2\pi m\sqrt{x})}{m^{\frac{3}{4}}} x^{\frac{5}{8}}.$$

Thus as x tends to infinity, the standard deviation σ of the number N of lattice points in a randomly located oval of this form and orientation satisfies

$$\sigma = \Omega(x^{\frac{3}{8}}).$$

This is of course larger than the order $O(x^{\frac{1}{4}})$ which has been established for σ when the oval is of class $\mathfrak{C}$.

Crudely one can say that the extreme flatness of the curve in the neighbourhood of a point of zero curvature implies the existence of an unduly large number of lattice points which for most locations of the centre of the oval will be all 'caught' or all 'missed'. Anyone who has had the experience of determining the area bounded by a closed curve, using the graphical method of 'counting squares', will be familiar with this phenomenon. It is in fact usual to guard against excessive errors of this sort by replacing exceptionally 'flat' arcs of the contour by approximating straight lines, and evaluating the corresponding portions of the area directly.

9. An application to graphical integration

There is, in fact, a practical application of these results which might on occasion be useful. The procedure of determining areas by 'counting squares' is exactly equivalent to the counting of lattice points, a square being counted as 'in' if and only if its centre (the associated lattice point) is within the bounding curve. Statistically this means that the random variable N is assumed to be equal to its mean value (the area of the oval expressed as a multiple of the area of a single small square), and so the standard deviation of the estimate of the area can for practical purposes be taken to be

$$a\sqrt{\left(\frac{L}{2\pi}\right)},$$

the coefficient of variation (the standard deviation expressed as a fraction of the mean) being therefore

$$\frac{a}{A}\sqrt{\left(\frac{L}{2\pi}\right)}, \tag{33}$$

where L is the perimeter and A is the area of the oval, assumed to be centrally symmetric. (For *general* ovals of class $\mathfrak{C}$, the expression (33) should be increased by a factor $\sqrt{2}$. This will also allow for the irregular fluctuations in σ, regarded as a function of x.)

Now, for a given value of A, L is least when the oval is circular; thus the process of determining areas by 'counting squares' has maximum accuracy if and only if the bounding curve is a circle, and for other curves of the same area the average error committed will be proportional to the square root of the perimeter.

The following table, which is based on the value $a = 0{\cdot}676497$, gives the coefficient of variation (as a percentage) for the determination of the area of a circle, and shows the rate at which the accuracy increases when a finer mesh is used. (The unit of length in the first two rows of the table is a single lattice step.)

Radius of circle:	5	10	20	40	
Area of circle:	79	314	1,257	5,027	
Coefficient of variation:	1·93	0·68	0·24	0·085	per cent.

If this method were employed to determine the value of π, an accuracy of one part in a thousand† would be obtained if the mesh used were fine enough for about 10,000 lattice points to be included within the circle.

It is important to notice that these results are applicable only to the graphical determination of the area contained within an oval of class $\mathfrak{C}$; they will *not* apply if the region to be measured is bounded in part by straight lines, for the curvature would be zero on these 'arcs' of the perimeter.

Appendix: the computation of the constant a

For the constant a, introduced in § 2 and defined by

$$a^2 = \frac{1}{2\pi^2}\sum_{l=1}^{\infty}\frac{r(l)}{l^{\frac{3}{2}}} = \frac{2}{\pi^2}\zeta(\tfrac{3}{2})L(\tfrac{3}{2}),$$

† i.e. a coefficient of variation of about 0·05 per cent.

the value $a = 0{\cdot}676497$ was determined by taking $\zeta(\frac{3}{2}) = 2{\cdot}612375$ and $L(\frac{3}{2}) = 0{\cdot}864503$. The existing tables of Dirichlet's L-function,

$$L(s) = 1^{-s}-3^{-s}+5^{-s}-7^{-s}+...,$$

are of no assistance here, and it was necessary to evaluate $L(\frac{3}{2})$ directly; the following method was found very convenient. If one writes

$$L(\tfrac{3}{2}) = 1-\frac{1}{3^{\frac{3}{2}}}+\frac{1}{5^{\frac{3}{2}}}-\sum_{m=2}^{\infty}\frac{1}{8m^{\frac{3}{2}}}\left\{\left(1-\frac{1}{4m}\right)^{-\frac{3}{2}}-\left(1+\frac{1}{4m}\right)^{-\frac{3}{2}}\right\},$$

the expression in brackets $\{\ \}$ can be expanded in powers of $1/m$ by the binomial theorem, and the resulting series, giving $L(\frac{3}{2})$ in terms of

$$\zeta(2{\cdot}5)-1,\quad \zeta(4{\cdot}5)-1,\quad ...,$$

is very rapidly convergent. The values of Riemann's zeta function were taken from the table of J. P. Gram, as quoted by H. B. Dwight.†

† *Mathematical Tables* (New York, 1941), 222.

REFERENCES

(i) *On Lattice-point Problems* (for general surveys see **6** and **9**).

1. H. Cramér, 'Über Zwei Sätze des Herrn G. H. Hardy', *Math. Zeits.* 15 (1922), 201–10.
2. G. H. Hardy, 'On the Expression of a Number as the Sum of Two Squares', *Quart. J. of Math.* 46 (1915), 263–83.
3. —— 'The Average Order of the Arithmetical Functions P(x) and $\Delta(x)$', *Proc. London Math. Soc.* (2), 15 (1916), 192–213.
4. —— and E. Landau, 'The Lattice Points of a Circle', *Proc. Royal Soc.* (A), 105 (1924), 244–58.
5. A. E. Ingham, 'On Two Classical Lattice Point Problems', *Proc. Cambridge Phil. Soc.* 36 (1940), 131–8.
6. E. Landau, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), bd. 2, 183–308.
7. L. K. Hua, 'The Lattice-points in a Circle', *Quart. J. of Math.* (Oxford), 13 (1942), 18–29.
8. E. C. Titchmarsh, 'The Lattice-points in a Circle', *Proc. London Math. Soc.* (2), 38 (1935), 96–115.
9. J. R. Wilton, 'The Lattice Points of a Circle: an Historical Account of the Problem', *Messenger of Math.* 58 (1929), 67–80.

(ii) *On Multiple Fourier Series.*

10. S. Bochner, 'Summation of Multiple Fourier Series by Spherical Means', *Trans American Math. Soc.* 40 (1936), 175–207.
11. G. H. Hardy, 'On Double Fourier Series, etc.', *Quart. J. of Math.* 37 (1906), 53–79.

12. E. W. Hobson, *The Theory of Functions of a Real Variable*, 2 (Cambridge, 1926, 2nd ed.), 698–719.

13. E. C. Titchmarsh, 'The Double Fourier Series of a Discontinuous Function', *Proc. Royal Soc.* (A), 106 (1924), 299–314.

(iii) *General.*

14. K. Chandra Sekharan, 'Bessel Summation of Series', *Proc. Indian Ac. of Science* (A), 17 (1943), 219–29.

15. M. Fréchet, *Recherches Théoriques Modernes sur la Théorie des Probabilités*, 1[er] livre (Paris, 1937), 26–7.

16. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), Theorem 127.

17. D. V. Widder, *The Laplace Transform* (Princeton, 1941), 115–18.

THE COMPACTING OF TOPOLOGICAL SPACES

By G. HIGMAN (*Manchester*)

[Received 26 March 1947]

1. Several constructions have been given for embedding a topological space S in a compact space.‡ In each case the space constructed is, in a sense, the largest compact space containing S as a subset everywhere dense; and it follows that the results of the constructions are homeomorphic. The object of this note is to show that two of these constructions, those due to Wallman (**4**) and to Gelfand and Silov (**6**), are in fact identical at a much lower level than that of the finished product, at least provided that S is normal.

The points of Wallman's space are the maximal dual-ideals§ in the lattice L of closed subsets of S; and his definition of a closed set amounts to saying that the closure of a set of maximal dual-ideals consists of all maximal dual-ideals containing the intersection of the dual-ideals of the given set. Gelfand and Silov consider as points the maximal ideals in the ring R of continuous, bounded, real-valued functions on S; and again the closure of a set of maximal ideals consists of all maximal ideals containing the intersection of the ideals of the given set. Below, I set up correspondences from the lattice of dual-ideals in L into the lattice of ideals in R, and vice versa; and I show that there is a subset of the lattice of dual-ideals in L which includes all maximal dual-ideals and their finite or infinite intersections, and similarly a subset of the lattice of ideals in R which includes all maximal ideals and their finite or infinite intersections, between which these correspondences are one-to-one and reciprocal, and within which they preserve intersections, finite or infinite. Plainly then, the correspondences set up a one-to-one correspondence between the two spaces which preserves the operation of closure; that is, they set up a homeomorphism. Moreover, the dual-ideal in L consisting of all closed sets containing a point a of S, and the ideal in R consisting of all functions which vanish at a correspond; and therefore the homeomorphism is the identity on S.

It is to be noticed that I have to impose on S the condition of

‡ Tychonoff (**2**), Čech (**3**), Wallman (**4**), Alexandroff (**5**), Gelfand and Silov (**6**). Cf. also Eilenberg (**8**).

§ For definitions see § 2.

normality, a stronger restriction than that of Gelfand and Silov, that S be completely regular. Wallman's construction, with slight modifications,‡ works for any topological space.

2. Let S be a normal T_1-space; that is, any single point of S is a closed set, and any two disjoint closed sets in S are contained in disjoint open sets. Let R be the ring of continuous, bounded, real-valued§ functions on S; and let L be the lattice of closed sets in S. By an ideal in R, we mean, of course, a subset of R containing with f also hf for every h in R, and with f and g also $f+g$. Similarly by a dual-ideal in L we mean a subset of L containing with A also $A \cup C$ for every C in L, and with A and B also $A \cap B$. A proper ideal in R is one not consisting either of the whole ring R, or of the function 0 alone; and similarly a proper dual-ideal in L is one not consisting either of the whole lattice L, or of the set S alone. A maximal ideal is a proper ideal that is not contained in any other proper ideal, and maximal dual-ideals are defined similarly.

If I is an ideal in R, we associate with it a set I^* of elements of L by the rule:

$A \in I^*$ if and only if, given any closed set F not meeting A, there exists f in I such that the lower bound of $|f|$ on F is positive.

Similarly, if J is a dual-ideal in L, we associate with it a set $J^\dagger$ of elements of R by the rule:

$f \in J^\dagger$ if and only if for every positive real number η there exists a set A in J on which $|f| < \eta$.

THEOREM 1. *I^* is a dual-ideal in L, and is proper if and only if I is proper.*

(i) If F does not meet $A \cup C$, then *a fortiori* it does not meet A. Hence $A \in I^*$ implies $A \cup C \in I^*$.

(ii) Assume that $A \in I^*$, $B \in I^*$, and let F be a closed set not meeting $A \cap B$. Then $F \cap B$ is a closed set not meeting A; hence there exists a function f_1 in I such that the lower bound η_1 of $|f_1|$ on $F \cap B$ is positive. Moreover, since $f_1^2 \in I$ if $f_1 \in I$, we may assume that f_1 is non-negative. Let F_1 be the set where $f_1 \leqslant \frac{1}{2}\eta_1$. Then $F \cap F_1$ is a closed set which does not meet B since, on $F \cap B$,

‡ Cf. Lefschetz (**7**), 20–1.

§ Only minor verbal changes would be necessary to deal with the ring of complex-valued functions.

$f_1 \geqslant \eta_1$. Hence there exists a function f_2 in I such that the lower bound η_2 of $|f_2|$ on $F \cap F_1$ is positive; and, as before, we may assume that f_2 is non-negative. Then $f_1+f_2 \in I$, and, on F,

$$|f_1+f_2| \geqslant \min(\tfrac{1}{2}\eta_1, \eta_2).$$

Hence $A \cap B \in I^*$.

This proves that I^* is a dual-ideal.

Now $(0)^* = (S)$; for the only closed set not meeting S is the null-set, on which every function has vacuously a positive lower bound, whereas, if $A \neq S$, there exists a point p not in A, and on (p) no function in (0) has positive lower bound, so that $A \mathrel{\bar{\in}} (0)^*$. Conversely $I^* = (S)$ implies $I = (0)$; for, if there exists f in I such that $|f(p)| = \eta \neq 0$, for any point p, then the set where $|f| \leqslant \frac{1}{2}\eta$ is not the whole space S, but belongs to I^*. Again $R^* = L$; since $1 \in R$, and 1 has positive lower bound on any closed set. Conversely $I^* = L$ implies $I = R$; for then the null-set is an element of I^*, so that there exists f in I such that $|f|$ has positive lower bound on S. Then $f^{-1} \in R$, $1 = ff^{-1} \in I$; so that $I = R$. That is, I^* is proper if and only if I is proper.

THEOREM 2. *$J^\dagger$ is an ideal in R, and is proper if and only if J is proper.*

(i) If $h \in R$, $|h| < \alpha$, then if $|f| < \eta/\alpha$ on A, $|hf| < \eta$ on A. Hence $f \in J^\dagger$ implies $hf \in J^\dagger$.

(ii) If $f \in J^\dagger$, $g \in J^\dagger$ and η is any positive real number, there exists A in J such that $|f| < \frac{1}{2}\eta$ on A, and B in J such that $|g| < \frac{1}{2}\eta$ on B. Then $A \cap B \in J$, and $|f+g| \leqslant |f|+|g| < \eta$ on $A \cap B$. Hence $(f+g) \in J^\dagger$.

This proves that $J^\dagger$ is an ideal in R.

Now obviously $(S)^\dagger = (0)$; for $f \in (S)^\dagger$ requires $|f| < \eta$ for all points in S and any positive number η. Conversely, if $J^\dagger = (0)$, then $J = (S)$; for, if $A \in J$ and $A \neq S$, then there exists a function f which is not identically zero but which is zero on A. Then $f \in J^\dagger$, so that $J^\dagger \neq (0)$. Again, $L^\dagger = R$; for the null-set is an element of L on which every function vacuously satisfies $|f| < \eta$, for any positive number η. Moreover, $J^\dagger = R$ implies $J^\dagger = L$; for then the function 1 belongs to $J^\dagger$, so that the null-set belongs to J. Hence $J^\dagger$ is proper if and only if J is proper.

THEOREM 3. *For a finite or infinite set of ideals* I_t, $(\bigcap_t I_t)^* \subset \bigcap_t I_t^*$.

Suppose that $A \in (\bigcap_t I_t)^*$. Then, if F is a closed set not meeting A, there exists f in $\bigcap_t I_t$ such that the lower bound of $|f|$ on F is positive. Then, for each t, $f \in I_t$. Hence $A \in I_t^*$ for each t; so that $A \in \bigcap_t I_t^*$, as required.

THEOREM 4. *For a finite or infinite set of dual-ideals* J_t, $(\bigcap_t J_t)^\dagger \subset \bigcap_t J_t^\dagger$.

Suppose that $f \in (\bigcap_t J_t)^\dagger$. Then, if η is a positive number, there exists a set A in $\bigcap_t J_t$ such that $|f| < \eta$ on A. Then $A \in J_t$ for each t. Hence $f \in J_t^\dagger$ for each t, or $f \in \bigcap_t J_t^\dagger$, as required.

COROLLARY (to Theorems 3 and 4). *If* $I_1 \subset I_2$, *then* $I_1^* \subset I_2^*$; *and, if* $J_1 \subset J_2$, *then* $J_1^\dagger \subset J_2^\dagger$.

For, if, for example, $I_1 \subset I_2$, then $I_1 = I_1 \cap I_2$. Therefore

$$I_1^* = (I_1 \cap I_2)^* \subset I_1^* \cap I_2^*.$$

That is, $I_1^* \subset I_2^*$, as required.

3. So far the correspondences $I \to I^*$ and $J \to J^\dagger$ have been considered separately. We now study their mutual relations.

THEOREM 5. $(I^*)^\dagger \supset I$.

For, if $f \in I$ and η is a positive number, let A be the set of points where $|f| \leqslant \frac{1}{2}\eta$. Then A is a closed set, and, on every closed set not meeting A, $|f| \geqslant \frac{1}{2}\eta$. Since $f \in I$, $A \in I^*$. But, on A, $|f| < \eta$. That is, given η, we can find A in I^* such that on A, $|f| < \eta$; that is, $f \in (I^*)^\dagger$.

THEOREM 6. $(J^\dagger)^* \supset J$.

Let $A \in J$, and let F be a closed set not meeting A. Because S is normal, there exists a function f such that $f = 0$ on A and $f = 1$ on F.‡ Then, since $f = 0$ on A and $A \in J$, we have $f \in J^\dagger$. But f has positive lower bound on F. That is, there exists a function in $J^\dagger$ having positive lower bound on any closed set not meeting A. That is, $A \in (J^\dagger)^*$.

We now make the definitions:

An ideal I *in* R *is cross-closed if* $(I^*)^\dagger = I$; *a dual-ideal* J *in* L *is cross-closed if* $(J^\dagger)^* = J$.

‡ By Urysohn's Lemma (**1**), of which partial cases have already been used in the argument.

THEOREM 7. *The following ideals and dual-ideals are cross-closed*: (i) (0), R, *and* (S), L; (ii) *all maximal ideals and maximal dual-ideals*; (iii) *the finite or infinite intersections of cross-closed ideals or cross-closed dual-ideals.*

Of these (i) is immediate from $(0)^* = (S)$, $(S)^\dagger = (0)$ and $R^* = L$, $L^\dagger = R$.

Again (ii) is almost a corollary of Theorems 5, 6, 1 and 2. For, if I is maximal, then, by Theorem 5, $(I^*)^\dagger \supset I$, so that either $(I^*)^\dagger = I$ or $(I^*)^\dagger = R$. But, by Theorems 2, 1, $(I^*)^\dagger = R$ implies in turn $I^* = L$ and $I = R$, which is false. Hence $(I^*)^\dagger = I$. Similarly, if J is maximal, $(J^\dagger)^* = J$.

To prove (iii), assume that I_t is cross-closed for every t. Then by Theorems 3 and 4, using also the corollary to them, we have

$$\{(\bigcap_t I_t)^*\}^\dagger \subset (\bigcap_t I_t^*)^\dagger \subset \bigcap_t (I_t^*)^\dagger = \bigcap_t I_t.$$

But the reverse inequality is Theorem 5, so that $\bigcap_t I_t$ is cross-closed. Similarly for dual-ideals.

THEOREM 8. *The correspondences $I \to I^*$ and $J \to J^\dagger$ are reciprocal one-to-one correspondences between the set of cross-closed ideals in R and the set of cross-closed dual-ideals in L which preserve finite or infinite intersections, under which maximal ideals correspond to maximal dual-ideals and vice versa, and under which the ideal of all functions that vanish at the point a corresponds to the dual-ideal of closed sets containing a.*

If I is cross-closed, then $I = (I^*)^\dagger$, and therefore $I^* = \{(I^*)^\dagger\}^*$, so that I^* is cross-closed. Similarly, if J is cross-closed, so is $J^\dagger$. However, if I and J are both cross-closed, the statements $I^* = J$ and $J^\dagger = I$ are obviously equivalent, whence the first statement of the theorem.

The second follows from Theorems 3 and 4. For, if each ideal I_t is cross-closed, and $I = \bigcap_t I_t$, $J = \bigcap_t I_t^*$, then $I^* \subset J$ by Theorem 3. By the corollary to Theorems 3 and 4, $I = (I^*)^\dagger \subset J^\dagger$. But by Theorem 4, $J^\dagger \subset I$. Hence $I = J^\dagger$ and $I^* = J$, as required.

Next, if I is maximal but I^* is not, there exists a maximal ideal J such that $I^* \subset J \subset L$, both inclusions being proper. Then by the corollary to Theorems 3 and 4, $I = (I^*)^\dagger \subset J^\dagger \subset R$, and by the fact that, for cross-closed ideals, the correspondence $J \to J^\dagger$ is one-to-one, the inclusions are again proper. This is in contradiction to the

fact that I is maximal. Hence I^* is maximal. Similarly, if J is maximal, $J^\dagger$ is maximal.

Finally, let $M(a)$ be the ideal in R of all functions that vanish at a. If F is a closed set not containing a, there exists a function f of R which vanishes at a and is equal to 1 on F. Then $f \in M(a)$. Hence (a), and therefore every closed set containing a, belongs to $\{M(a)\}^*$. However, if B is a closed set not containing a, then (a) is a closed set not meeting B on which no function of $M(a)$ has positive lower bound, so that $B \bar{\in} \{M(a)\}^*$. That is, $\{M(a)\}^*$ contains all closed sets containing a and only those.

This concludes the proof of Theorem 8, and completes the programme laid down in the introduction.

REFERENCES

1. P. Urysohn, 'Über die Mächtigkeit der zusammenhängenden Mengen', *Math. Annalen*, 94 (1925), 262–95.
2. A. Tychonoff, 'Über die topologische Erweiterung von Räumen', ibid. 102 (1930), 544–61.
3. E. Čech, 'On bicompact spaces', *Annals of Math.* 38 (1937), 823–44.
4. H. Wallman, 'Lattices and topological spaces', ibid. 39 (1938), 112–26.
5. P. Alexandroff, 'Bikompakte Erweiterungen topologischer Räume', *Recueil Math.* (Mat. Sbornik) 5 (1939), 403–23.
6. I. Gelfand and G. Silov, 'Über verschiedene Methoden der Topologisierung...', ibid. 9 (1941), 25–39.
7. S. Lefschetz, *Algebraic Topology* (New York, 1942).
8. S. Eilenberg, 'Banach Space methods in Topology', *Annals of Math.* 43 (1942), 568–79.

REMARKS ON AHLFORS' DISTORTION THEOREM

By W. K. HAYMAN (*Cambridge*)

[Received 15 March 1947]

1. We consider in the plane of $s = \sigma + i\tau$ a simply-connected domain G excluding $s = \infty$. Let

$$S_1 = \Sigma_1 + iT_1, \qquad S_2 = \Sigma_2 + iT_2 \qquad (-\infty \leqslant \Sigma_1 < \Sigma_2 \leqslant +\infty)$$

be two accessible boundary points of G. Each straight line $\mathrm{Rl}\, s = \sigma$ $(\Sigma_1 < \sigma < \Sigma_2)$ meets G in one or more segments Q each of which forms a cross-cut in G and so divides G into two simply-connected domains whose points are separated by Q.

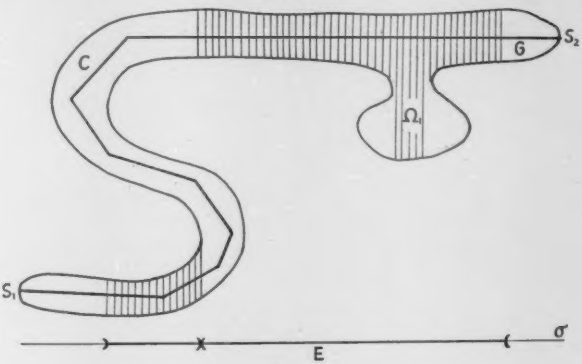

Fig. 1.

We now connect S_1, S_2 by a cross-cut C, given by

$$s = s(t) \quad (0 \leqslant t \leqslant 1), \qquad s(0) = S_1, \qquad s(1) = S_2.$$

Without loss of generality we may suppose that C consists of a finite or enumerable number of straight-line segments none of which is parallel to the axis and whose end-points have no limit points except possibly S_1, S_2.

Since C meets each straight line $\mathrm{Rl}\, s = \sigma$ $(\Sigma_1 < \sigma < \Sigma_2)$ in an odd number of points, the same is true of one at least of those segments of this line which lie in G. Such a segment must separate S_1, S_2. We denote the first such segment, starting from S, which C meets by θ_σ and its length by $\theta(\sigma)$. Then Ahlfors† proved

† L. Ahlfors, *Acta Soc. Sci. Fenn.* (Nova Ser.), No. 9.

THEOREM A. *Suppose that G is mapped one to one and conformally on the strip $0 < \eta < a$ in the plane of $\zeta = \xi + i\eta$ so that $s = S_1, S_2$ correspond to $\xi = -\infty, +\infty$ respectively. Let $\xi_1(\sigma)$, $\xi_2(\sigma)$ be the least and greatest values of ξ on θ_σ, for $\Sigma_1 < \sigma < \Sigma_2$. Then, if*

$$\Sigma_1 < \sigma_1 < \sigma_2 < \Sigma_2$$

and

$$I = \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)} > 2,$$

we have

$$\xi_1(\sigma_2) - \xi_2(\sigma_1) \geqslant a(I-4).$$

In this paper I obtain the greatest lower bound of $\xi_1(\sigma_2) - \xi_2(\sigma_1)$ for given

$$I = \int_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)} \quad (0 < I < \infty),$$

under the hypotheses of Theorem A. I also prove a parallel theorem in which best possible bounds for distortion are obtained in terms of I for another class of mappings of G. Theorem A has been extensively used for investigating the order of growth of regular functions. Ahlfors† himself employed the result to prove the Denjoy hypothesis that an integral function of order ρ has at most $[2\rho]$ asymptotic values. Cartwright‡ employed Theorem A to prove that a function regular and p-valent in $|z| < 1$ is at most of order $[1-|z|]^{-2(p+1)}$ and the method was extended by Spencer§ to mean p-valent functions.

2. I consider instead of the domain G of Theorem A an open set Ω which meets any line $\mathrm{Rl}\, s = \sigma$ in at most one finite segment θ_σ of length $\theta(\sigma)$. The set of σ for which $\theta(\sigma) > 0$ we denote by E. Since G is an open plane set, E is an open linear set and consists of a finite or enumerable number of open intervals, and $\theta(\sigma)$ is lower-semi-continuous in E. It follows that

$$I = \int_E \frac{d\sigma}{\theta(\sigma)} \tag{2.1}$$

exists as a finite or infinite Lebesgue integral. I shall suppose in the sequel that I is finite.

† L. Ahlfors, *Acta Soc. Sci. Fenn.* (Nova Ser.), No. 9.
‡ M. L. Cartwright, *Math. Annalen*, 111 (1935), 98–114.
§ D. C. Spencer, *Trans. American Math. Soc.* (1940), 48, 418–33.

Consider the particular case where E consists of the single segment $0 < \sigma < K$ and $\tau_1(\sigma) = 0$, $\tau_2(\sigma) = K'$. Then

$$\theta(\sigma) = K', \qquad 0 < \sigma < K, \qquad I = \frac{K}{K'}.$$

If we denote the points $s = 0, K, K+iK', iK'$ by A, B, C, D, then Ω consists of the rectangle $ABCD$ in this case. Suppose that

$$\zeta = \xi + i\eta = \zeta_0(S)$$

maps $ABCD$ on the strip $0 < \eta < 1$ in the ζ-plane in such a manner that A, C correspond respectively to $\xi = -\infty, +\infty$. Suppose that B, D map into $\zeta = \xi_1 + 0i$, $\xi_2 + i$ respectively. Then

$$\xi_0(I) = \xi_1 - \xi_2$$

is independent of the particular mapping chosen and depends only on I. I shall show that $\xi_0(I)$ gives the exact upper bound in Ahlfors' theorem.

In order to prove another distortion theorem, consider another particular mapping. Let D_0 be the domain consisting of all points in $|z| < 1$ except two slits from -1 to 0 and from ρ to 1 $(0 < \rho < 1)$, along the real z-axis. Suppose that ρ is so chosen that we can map $ABCD$ on D_0 in such a manner that A, D; B, C correspond respectively to $z = -1, +1$, so that the sides AD, BC map into the slits, while AB, CD map into $|z| = 1$. This will be possible if ρ is a certain function,

$$\rho = \rho_0(I),$$

of I. The functions $\rho_0(I)$, $\xi_0(I)$ are related to elliptic functions and will be investigated in detail in the last three sections of the paper.

3. The results may now be stated as follows.

THEOREM I. *Let Ω be the set defined at the beginning of § 2 and suppose that*

$$\zeta = \xi + i\eta = \zeta(s)$$

maps Ω one to one and conformally on a set O lying in the strip $0 < \eta < a$.

Let γ_σ be the image of θ_σ by $\zeta(s)$ and suppose that for σ in E, γ_σ has limit points on $\eta = 0$ and $\eta = a$. Let $\xi_1(\sigma)$, $\xi_2(\sigma)$ be the lower and upper bounds of ξ on γ_σ, and let $\xi_1(E)$, $\xi_2(E)$ be the upper and lower bounds of $\xi_1(\sigma)$ and $\xi_2(\sigma)$ respectively for σ in E. Then, if I, defined by (2.1), *is finite, we have*

$$\xi_1(E) - \xi_2(E) \geqslant a\xi_0(I).$$

THEOREM II. *Under the hypotheses of Theorem A, we have*

$$\xi_1(\sigma_2)-\xi_2(\sigma_1) > a\xi_0(I)$$

and this bound is exact for each I $(0 < I < \infty)$.

THEOREM III. *Let Ω be the set of Theorem I and let $z = z(s)$ be 'schlicht' in Ω and satisfy $|z(s)| < 1$. Let g_σ be the image of θ_σ by $z(s)$, and suppose that the points $z = 0$, ρ are separated† in $|z| < 1$ by each of the g_σ, σ in E. Then we have $\rho \geqslant \rho_0(I)$.*

As an immediate corollary we deduce

THEOREM IV. *Let Ω be the set of Theorem I and let $z = z(s)$ be 'schlicht' in Ω and have all its values lying in a simply-connected domain D. Let g_σ be the image of θ_σ by $z(s)$ and suppose that the points $z = z_1, z_2$ are separated† in D by each of the g_σ. Then*

$$d[z_1, z_2; D] \geqslant d_0(I) = \tfrac{1}{2}\log\frac{1+\rho_0(I)}{1-\rho_0(I)},$$

where $d[z_1, z_2; D]$ is the hyperbolic distance of z_1, z_2 with respect to D.

The hyperbolic distance is defined as follows. We map D one to one and conformally on the unit circle $|t| < 1$ so that $z = z_1, z_2$ map into $t = 0, t_2$. Then we define

$$d[z_1, z_2; D] = \tfrac{1}{2}\log\frac{1+|t_2|}{1-|t_2|}.$$

The numerical nature of the functions $\xi_0(I)$, $\rho_0(I)$, $d_0(I)$ is investigated in the last three sections. The principal results may be put together in

THEOREM V. *Consider a system of Jacobian elliptic functions with quarter-periods K, iK', where $K/K' = I$. Let k, ik' be the modulus and complementary modulus of the system. Then*

$$\xi_0(I) = \frac{2}{\pi}\log\frac{k}{k'}, \tag{3.1}$$

$$\rho_0(I) = k, \tag{3.2}$$

$$d_0(I) = \tfrac{1}{2}\log\frac{1+k}{1-k}. \tag{3.3}$$

† We say that a set C *separates* two points P_1, P_2 in a domain D if every Jordan arc contained in D and having P_1, P_2 as end-points meets C.

We have the inequalities

$$\frac{4}{\pi}\log 2-\frac{1}{I} < \xi_0(I) \leqslant 1-\frac{1}{I} \quad (0 < I \leqslant 1), \tag{3.4}$$

$$I-1 \leqslant \xi_0(I) < I-\frac{4}{\pi}\log 2 \quad (1 \leqslant I < \infty), \tag{3.5}$$

$$\frac{1}{\sqrt{2}}\exp\frac{\pi}{2}\left(1-\frac{1}{I}\right) \leqslant \rho_0(I) < 4\exp\left(-\frac{\pi}{2I}\right) \quad (0 < I \leqslant 1), \tag{3.6}$$

$$1-8e^{-\pi I} < \rho_0(I) \quad (0 < I < \infty), \tag{3.7}$$

$$\frac{\pi I}{2}-\log 2 < d_0(I) \quad (0 < I < \infty), \tag{3.8}$$

$$d_0(I) \leqslant \frac{\pi I}{2}-\frac{\pi}{2}+\log(1+\sqrt{2}) \quad (1 \leqslant I < \infty), \tag{3.9}$$

in which all the constants are 'best possible' and the bounds are attained either when $I = 1$ or in the limit when $I \to 0$ or $I \to \infty$.

4. In this section I deduce Theorem II from Theorem I and give some applications of Theorems III and IV. In the remainder of the paper I shall prove Theorems I, III, and V in turn.

I use the notation of the first section. The segment θ_σ in Theorem A is the first segment of $\mathrm{Rl}\, s = \sigma$, starting from S_1, in the domain G which meets C and separates S_1, S_2 in G. Let P_σ be the first point of C which θ_σ meets. Then it is clear that the points P_σ are in order of increasing σ along C. It follows that, at each discontinuity of P_σ, P_σ moves to a segment of C further away from S_1 as σ increases, so that P_σ cannot have more discontinuities than there are segments of C. If we define Ω to be the set of all θ_σ $(\sigma_1 < \sigma < \sigma_2)$ for which P_σ is continuous, Ω is clearly an open set and the mapping of Ω on part of the ζ-plane satisfies the hypotheses of Theorem I. By applying that theorem we deduce the inequality of Theorem II with $\geqslant$ instead of $>$. To prove the strict inequality it will be sufficient to verify that the extremal mappings which yield equality in Theorem I do not satisfy the hypotheses of Theorem A. This will appear in the proof of Theorem I.

To prove that the bound in Theorems I and II is exact, consider the extremal mapping of the rectangle $ABCD$ of § 2 on the strip $0 < \eta < 1$ in the ζ-plane, where $AB = K$, $BC = K'$ and take $\sigma_1 = \delta$, $\sigma_2 = K-\delta$. The vertices A, C correspond to $\xi = -\infty, +\infty$ respectively, and B, D map into $\zeta = \xi_1+0i$, ξ_2+i respectively.

It follows that the images of the segments θ_{σ_1}, θ_{σ_2} contain some points for which $\xi > \xi_2$ and some for which $\xi < \xi_1$. Hence

$$\xi_2(\sigma_1) > \xi_2, \qquad \xi_1(\sigma_2) < \xi_1,$$

so that
$$\xi_1(\sigma_2) - \xi_2(\sigma_1) < \xi_1 - \xi_2 = \xi_0\left(\frac{K}{K'}\right).$$

By choosing δ sufficiently small we can make

$$I = \int\limits_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)}$$

arbitrarily near to K/K', so that for given a, I and $\epsilon > 0$ we can make $\xi_1(\sigma_2) - \xi_2(\sigma_1)$ less than $a\{\xi_0(I)+\epsilon\}$ under the hypotheses of Theorem A. This shows that the bound in Theorem II is exact.

As an example of the use of Theorem IV we may consider the following problem. Let $\mathfrak{R}$ be a simply-connected Riemann surface over the plane of $s = \sigma + i\tau$ and suppose that no 'schlicht' segment of $\mathfrak{R}$ over the line $\mathrm{Rl}\, s = \sigma$ has a length greater than $\theta(\sigma)$ $(0 < \theta(\sigma) < \infty)$. Let P_1, P_2 over $s_1 = \sigma_1 + i\tau_1$, $s_2 = \sigma_2 + i\tau_2$ respectively, be two points of $\mathfrak{R}$. Let us consider the hyperbolic distance $d\{P_1, P_2; \mathfrak{R}\}$ which is defined as follows. Suppose that $\mathfrak{R}$ is mapped one to one and conformally on the unit circle $|t| < 1$, so that P_1, P_2 become $t = 0$, t_2 respectively. Then

$$d[P_1, P_2; \mathfrak{R}] = \tfrac{1}{2}\log\frac{1+|t_2|}{1-|t_2|}.$$

Making use of Theorem IV we can prove

THEOREM VI. *If P_1, P_2 are two points of $\mathfrak{R}$ over $s_1 = \sigma_1 + i\tau_1$, $s_2 = \sigma_2 + i\tau_2$ respectively and*

$$\int\limits_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)} = I > 0,$$

then
$$d[P_1, P_2; \mathfrak{R}] \geqslant d_0(I) > \tfrac{1}{2}\pi I - \log 2.$$

To prove Theorem VI, we map $\mathfrak{R}$ one to one and conformally on $|t| < 1$ so that P_1, P_2 correspond to $t = 0$, ρ respectively. We may join P_1, P_2 by a polygonal arc in $\mathfrak{R}$ which consists of a finite number of segments none of which are parallel to the τ-axis. For $\sigma_1 < \sigma < \sigma_2$, let θ_σ be the first segment of $\mathrm{Rl}\, s = \sigma$ which C meets an odd number of times, starting from P_1. If we omit the finite set of σ for which $\mathrm{Rl}\, s = \sigma$ contains a vertex of C, the remaining σ form an open set E, and the corresponding θ_σ an open set Ω. To the mapping of Ω which

is obtained in this way, we can then apply Theorem IV and we obtain Theorem VI.

As a further application we can deduce

THEOREM VII. *Suppose that $f(z)$ is regular in $|z| < 1$ and omits some value w_R on every circle $|w| = R \geqslant 0$. Then*

$$|f(\rho e^{i\theta})| < 16\,|f(0)|\left(\frac{1+\rho}{1-\rho}\right)^2.$$

We consider the Riemann surface $\mathfrak{R}$ of $s = \log f(z)$. Since $f(z)$ omits some value on every circle, the variation of τ on any segment $\mathrm{Rl}\, s = \sigma$ which lies entirely in $\mathfrak{R}$ is at most 2π. Hence we may apply Theorem VI with $\theta(\sigma) \leqslant 2\pi$, taking for P_1, P_2 the points in $\mathfrak{R}$ corresponding to $z = 0$, $\rho e^{i\theta}$ respectively. Then we have

$$\sigma_1 = \log|f(\sigma)|, \qquad \sigma_2 = \log|f(\rho e^{i\theta})|,$$

and

$$I = \int\limits_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)} \geqslant \frac{\sigma_2-\sigma_1}{2\pi},$$

so that Theorem VI yields

$$\tfrac{1}{2}\log\frac{1+\rho}{1-\rho} = d[P_1, P_2;\, \mathfrak{R}] > \frac{\sigma_2-\sigma_1}{4} - \log 2,$$

$$\sigma_2 < \sigma_1 + 4\log 2 + 2\log\frac{1+\rho}{1-\rho},$$

which proves Theorem VII.

When the values w_R in Theorem VII fill the negative real axis, and $f(0)$ is real, $f(z)$ is subordinate to $f(0)\{(1+z)/(1-z)\}^2$, and we may replace 16 by 1 and the $<$ sign by $\leqslant$. It is of interest whether this stronger result still holds under the more general hypotheses of Theorem VII. More generally we may ask whether, if $\theta_\sigma \leqslant a$ in Theorem VI, we may conclude that

$$d[P_1, P_2;\, \mathfrak{R}] \geqslant \frac{\pi|\sigma_2-\sigma_1|}{2a}$$

with equality only in the case of a strip of width a. The method here developed does not seem sufficiently powerful to prove this and related best-possible results.

5. We have seen in the last section how Theorem II follows from Theorem I. Theorem IV follows immediately from Theorem III on mapping the domain D of Theorem IV one to one and conformally

on the unit circle. It therefore remains to prove Theorems I, III, and V.

The plan of the rest of the paper will be as follows. In this section we introduce Lemma 1, which is fundamental to the proofs of Theorems I and III, and is a slight modification of the principle used by Ahlfors. Then §§ 6–8 will be occupied with the proof of Theorem I and §§ 9, 10 with that of Theorem III. In §§ 11–13 I investigate the nature of the functions $\xi_0(I)$, $\rho_0(I)$, and $d_0(I)$, and prove Theorem V.

LEMMA 1. *Let Ω be the open set of Theorem I, and suppose that $u = u(s)$ is regular and 'schlicht' in Ω. Let $l(\sigma)$ be the length of the transform of θ_σ by $u(s)$, and let A be the total area of the transform of Ω by $u(s)$. Then we have*

$$\int_E \frac{l^2(\sigma)\, d\sigma}{\theta(\sigma)} \leqslant A.$$

Let $\alpha < \sigma < \beta$ be an open interval belonging to E. Then

$$l^2(\sigma) = \left[\int_{\theta_\sigma} |u'(s)|\, d\tau\right]^2 \leqslant \int_{\theta_\sigma} d\tau \int_{\theta_\sigma} |u'(s)|^2\, d\tau$$

by Schwarz's inequality. Hence

$$\frac{l^2(\sigma)}{\theta(\sigma)} \leqslant \int_{\theta_\sigma} |u'(s)|^2\, d\tau.$$

Integrating over $\alpha < \sigma < \beta$ we have

$$\int_\alpha^\beta \frac{l^2(\sigma)\, d\sigma}{\theta(\sigma)} \leqslant \int_\alpha^\beta d\sigma \int_{\theta_\sigma} |u'(s)|^2\, d\tau.$$

Adding for the separate intervals of E, we have

$$\int_E \frac{l^2(\sigma)\, d\sigma}{\theta(\sigma)} \leqslant \int_\Omega |u'(s)|^2\, d\sigma d\tau = A,$$

which proves the lemma. We deduce

LEMMA 2. *Under the hypotheses of Theorem I, the variation of $\xi = \mathrm{Rl}\,\zeta$ in Ω is not less than aI.*

Suppose that the lemma is false. Then the function $\zeta = \zeta(s)$ of Theorem I is 'schlicht' in Ω, and the values taken by the function all lie in a rectangle of area less than a^2I. We may apply Lemma 1 to the mapping. The transforms γ_σ of θ_σ have limit points both on $\eta = 0$ and $\eta = a$, and so their length is at least a. Applying the

formula of Lemma 1 we obtain

$$a^2 \int\limits_E \frac{d\sigma}{\theta(\sigma)} = a^2 I < a^2 I,$$

a contradiction, which proves the lemma.

The proof of our results will be as follows. Let γ_σ be the transform of θ_σ by the function $\zeta(s)$ of Theorem I. I shall define a conformal transformation of the set of γ_σ into a rectangle of sides K, K' in such a manner that each transform of a γ_σ joins the pair of opposite sides of length K. With the hypotheses of Theorem I, if

$$\xi_1(E) - \xi_2(E) < a\xi_0(I),$$

I shall show that this mapping is possible in such a way that

$$\frac{K}{K'} < I,$$

thus giving a contradiction to Lemma 2. This will prove Theorem I. I proceed similarly to prove Theorem III.

6. Proof of Theorem I

We have

LEMMA 3. *Let γ_σ be the image of the segment θ_σ by the function $\zeta(s)$ of Theorem I and let $\bar{\gamma}_\sigma$ be the closure of γ_σ in the ζ-plane. Then there exist two continua, $\gamma_{(1)}$, stretching from $\xi = +\infty$ to $\xi = \xi_1(E)$, and $\gamma_{(2)}$, stretching from $\xi = \xi_2(E)$ to $\xi = -\infty$ which are both contained in $0 \leqslant \eta \leqslant a$ and do not contain a point of any $\bar{\gamma}_\sigma$.*

I show first that, if σ is contained in the set E of Theorem I for $\sigma_1 < \sigma < \sigma_2$, then $\bar{\gamma}_{\sigma_1}$ and $\bar{\gamma}_{\sigma_2}$ have no common points. In fact, since $\zeta(s)$ is 'schlicht' in Ω, it follows that γ_{σ_1} and γ_{σ_2} have no common point. Suppose now that γ_{σ_1}, γ_{σ_2} have a common limit point $\zeta = \xi_0$ on $\eta = 0$. Since $\zeta(s)$ is regular and 'schlicht' in Ω $(\sigma_1 < \sigma < \sigma_2)$, γ_σ separates γ_{σ_1} and γ_{σ_2}. Hence γ_σ has a limit point at $\zeta = \xi_0$ $(\sigma_1 < \sigma < \sigma_2)$. Given $\epsilon > 0$, we can map the strip $0 < \eta < a$ on a rectangle $0 < x < \epsilon$, $0 < y < 1$ conformally so that the line $\eta = a$ maps into $y = 1$, and $\zeta = \xi_0$ maps into a point on $y = 0$. We obtain an induced mapping

$$z = x + iy = z[\zeta(s)]$$

of the θ_σ for $\sigma_1 < \sigma < \sigma_2$ on this rectangle so that

$$\operatorname*{l.u.b.}_{\theta_\sigma} y(\sigma + i\tau) = 1, \qquad \operatorname*{g.l.b.}_{\theta_\sigma} y(\sigma + i\tau) = 0.$$

If we take

$$\epsilon < \int\limits_{\sigma_1}^{\sigma_2} \frac{d\sigma}{\theta(\sigma)}$$

and take for Ω in Lemma 2 the points on the θ_σ for $\sigma_1 < \sigma < \sigma_2$, we obtain a contradiction to that lemma. This shows that $\gamma_{\sigma_1}, \gamma_{\sigma_2}$ cannot have a common limit point on $\eta = 0$, nor similarly on $\eta = a$. Thus $\bar{\gamma}_{\sigma_1}, \bar{\gamma}_{\sigma_2}$ have no common points.

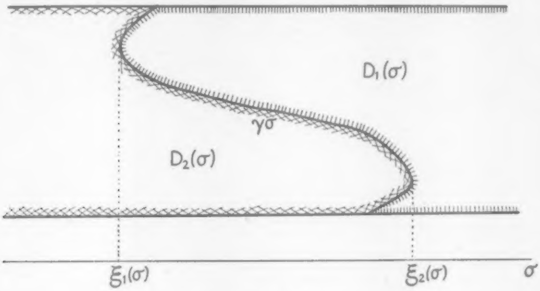

FIG. 2.

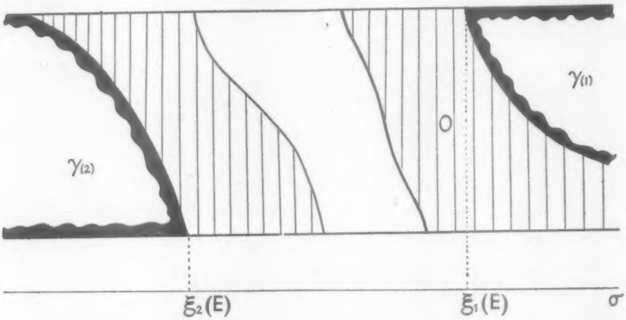

FIG. 3.

The curves γ_σ determine two domains $D_1(\sigma)$, $D_2(\sigma)$ in $0 < \eta < a$, having respectively $\xi = +\infty, -\infty$ as boundary points. Since $\gamma_{\sigma_1}, \gamma_{\sigma_2}$ have no common points, the sets of domains $D_1(\sigma)$ for σ in E are monotonic, i.e. of any two $D_1(\sigma)$ one is contained in the other. Hence the same is true of the sets of closures $\overline{D_1(\sigma)}$ and similarly $\overline{D_2(\sigma)}$, which are non-null continua. If we form the products

$$\gamma_{(1)} = \prod_{\sigma \text{ in } E} \overline{D_1(\sigma)}, \qquad \gamma_{(2)} = \prod_{\sigma \text{ in } E} \overline{D_2(\sigma)},$$

then $\gamma_{(1)}, \gamma_{(2)}$ have the properties stated in the lemma.

We consider the strip $-\infty < \xi < +\infty$, $0 \leqslant \eta \leqslant a$ made compact by the introduction of the different boundary points $\xi = -\infty, +\infty$. Then, being products of monotonic sets of non-null continua, $\gamma_{(1)}, \gamma_{(2)}$ are themselves non-null continua.† Also each $\bar{D}_1(\sigma)$ meets the line $\xi = \xi_1(E)$ and the point $\xi = +\infty$, and hence so does $\gamma_{(1)}$. Similarly $\gamma_{(2)}$ stretches from $\xi = \xi_2(E)$ to $\xi = -\infty$. Finally, if σ is contained in E and ϵ is sufficiently small, so are $\sigma-\epsilon$, $\sigma+\epsilon$. Then $\bar{\gamma}_{\sigma-\epsilon}, \bar{\gamma}_{\sigma+\epsilon}$ have no point in common with γ_σ, as was proved above, and so either $\bar{D}_1(\sigma-\epsilon)$ and $\overline{D_2(\sigma+\epsilon)}$ do not meet $\bar{\gamma}_\sigma$ or $\overline{D_2(\sigma-\epsilon)}$ and $\overline{D_1(\sigma+\epsilon)}$ do not meet $\bar{\gamma}_\sigma$. In either case we see that $\gamma_{(1)}$ and $\gamma_{(2)}$ do not meet $\bar{\gamma}_\sigma$, and this completes the proof of the lemma.

7. My proof of Theorem I depends on the solution of a certain extremal problem in the mapping of doubly-connected domains.

LEMMA 4. *Let D_0 be the doubly-connected domain consisting of all points in the complex u-plane except the points $-\infty \leqslant u \leqslant r_0$, $0 \leqslant u \leqslant 1$, on the real axis. Let D be a doubly-connected domain in the v-plane whose complement consists of a continuum Γ_1 joining $v = 0, 1$ and a continuum Γ_2 stretching from $|v| = r$ to infinity. Then, if D can be mapped one to one and conformally on D_0, we have $r > r_0$, unless D coincides with D_0.*

A related problem was solved by Schiffer,‡ who considered extremal mappings of doubly-connected domains whose boundary consisted of two continua joining two pairs of given points. My result is simpler than Schiffer's, however, and does not require the full strength of his variational method.

Following Schiffer,‡ we remark that under the hypotheses of Lemma 4, r certainly possesses a positive lower bound, and that there exists a domain D for which this lower bound is attained. In fact the mapping functions which map D_0 on the domains D form a compact family in D_0. It is therefore sufficient to show that, if D_1 is a domain other than D_0 which satisfies the hypotheses for D in Lemma 4, and if r_1 is the corresponding value of r, then D_1 can be mapped one to one and conformally on a domain D_2 also satisfying these hypotheses with $r = r_2 < r_1$. The proof of this depends on the Bieberbach theory of 'schlicht' functions.

† See, for example, F. Hausdorff, *Mengenlehre* (New York, 1944, 3rd ed.), chap. VI, 163.

‡ M. Schiffer, *Quart. J. of Math.* (*Oxford*), 17 (1946), 197–213.

Let Γ_1^1, Γ_2^1 be the complementary continua of D_1. Let D_1^* be the simply-connected domain, containing the origin, whose complement consists of Γ_2^1, which we suppose to contain some points not on the negative real axis. A similar argument applies if Γ_1^1 is not confined to the real axis. Let

$$v_1 = \psi_1(t) = a_1 t + a_2 t^2 + \ldots$$

map $|t| < 1$ one to one and conformally on D_1^*. Since D_1^* does not contain all points of $|v_1| = r_1$, we have†

$$|a_1| \leqslant 4r_1.$$

Let t_0 be the point such that $\psi(t_0) = 1$. Then we have‡

$$1 = \psi_1(t_0) < \frac{|a_1||t_0|}{[1-|t_0|]^2} \leqslant \frac{4r_1|t_0|}{[1-|t_0|]^2}$$

since Γ_2^1 is not confined to the negative real axis. Without loss of generality we suppose t_0 to be real and positive and define

$$r_2 = \frac{(1-t_0)^2}{4t_0} < r_1.$$

Then
$$v_2 = \psi_2(t) = \frac{4r_2 t}{(1-t)^2}$$

maps $|t| < 1$ on a simply-connected domain D_2^* whose complement consists of the straight-line segment Γ_2^2 $(-\infty < v_2 < -r_2)$ of the real v_2-axis. Also $\psi_2(t_0) = 1$, so that $v_2 = \psi_2\{t[v_1]\}$ maps the complementary continuum Γ_1^1 of D_1 on a continuum joining $v_2 = 0, 1$. Thus

$$v_2 = \psi_2\{t[v_1]\}$$

maps D_1 on a domain D_2 satisfying the conditions for D of Lemma 4, with $r = r_2 < r_1$. This completes the proof of the lemma. We deduce

LEMMA 5. *Let $\gamma_{(1)}$, $\gamma_{(2)}$ be the continua of Lemma 3 and let Δ be the domain consisting of all points of the strip $0 < \eta < a$ which are not on $\gamma_{(1)}$ or $\gamma_{(2)}$. Let*

$$\lambda = \mu + i\nu = \lambda(\zeta)$$

map Δ on the strip $0 < \nu < 1$, one to one and conformally, so that the boundary points of Δ on $\gamma_{(1)}$, $\gamma_{(2)}$, map into points on the half-lines $\mu \geqslant \mu_0$, $\nu = 1$ and $\mu \leqslant 0$, $\nu = 0$ respectively and the remaining boundary points of Δ on the lines $\eta = 0$, a map into the half-lines $\mu > 0$, $\nu = 0$ and $\mu < \mu_0$, $\nu = 1$ respectively. Then

$$\xi_1(E) - \xi_2(E) \geqslant a\mu_0.$$

† L. Bieberbach, *Funktionentheorie*, vol. ii (Berlin, 1931), 75.

‡ L. Bieberbach, op. cit. 77.

We put
$$u = e^{\pi\lambda}, \tag{7.1}$$
$$v = e^{\pi\zeta/a}. \tag{7.2}$$

Then v is a regular 'schlicht' function of u for $\operatorname{Im} u > 0$ and, on the segment $1 < u < \infty$ of the positive real u-axis, v is continuous and takes real and positive values; on the segment $-e^{\pi\mu_0} < u < 0$, v is continuous and takes real and negative values. If we put
$$v = \psi(u), \qquad \psi(\bar{u}) = \overline{\psi(u)},$$
$v = \psi(u)$ is a regular 'schlicht' function of u in the domain D_0 of Lemma 4 with $r_0 = e^{\pi\mu_0}$.

Since $\bar{\gamma}_\sigma$ contains no point of $\gamma_{(1)}$ or $\gamma_{(2)}$, $v = \psi(u)$ takes no value which lies on the transforms of these continua by (7.2). These transforms are continua stretching from $v = v_1$ to $v = \infty$ and from $v = 0$ to $v = v_2$ respectively, where
$$|v_1| = e^{(\pi/a)\xi_1(E)}, \qquad |v_2| = e^{(\pi/a)\xi_2(E)}.$$

Hence the function
$$\psi_2(v) = \frac{\psi(u)}{v_2}$$
gives a 'schlicht' mapping of D_0 on a domain satisfying the conditions for D of Lemma 4 with
$$r = \left|\frac{v_1}{v_2}\right| = e^{(\pi/a)[\xi_1(E)-\xi_2(E)]}.$$

Applying Lemma 4, we obtain
$$e^{(\pi/a)[\xi_1(E)-\xi_2(E)]} \geqslant e^{\pi\mu_0},$$
$$\xi_1(E)-\xi_2(E) \geqslant a\mu_0$$
which proves Lemma 5.

8. We can now prove Theorem I. Let μ_0 be the μ_0 of Lemma 5, and let I_0 be defined by
$$\xi_0(I_0) = \mu_0,$$
where $\xi_0(I)$ is the function defined in § 2. Then we can map the strip $0 < \nu < 1$ in the λ-plane on the rectangle $\mathfrak{R}$ $(0 < x < I_0;\ 0 < y < 1)$ in the z-plane in such a manner that $\lambda = 0,\ +\infty,\ \mu_0+i,\ -\infty$ correspond to $z = 0,\ I_0,\ I_0+i,\ i$, respectively. It then follows from Lemmas 3, 5 that
$$z = z\{\lambda[\xi(s)]\}$$
gives a 'schlicht' mapping of the set Ω of Theorem I into the rectangle $\mathfrak{R}$ in such a manner that the transforms of the segments θ_σ

have limit points on both $y = 0$ and $y = 1$. Then Lemma 2 gives $I_0 \geqslant I$ and hence $\mu_0 \geqslant \xi_0(I)$. Then Lemma 5 gives

$$\xi_1(E) - \xi_2(E) \geqslant a\mu_0 \geqslant a\xi_0(I),$$

which proves Theorem I.

We note that the extremal mapping in Lemma 4 corresponds to the case when $\gamma_{(1)}$, $\gamma_{(2)}$ form half-lines on the boundary of the strip $0 < \eta < a$ in the ζ-plane. Then $\xi_1(E)$, $\xi_2(E)$ are the ξ-coordinates of the finite end-points of $\gamma_{(1)}$, $\gamma_{(2)}$ respectively. Such a mapping does not satisfy the hypotheses of Ahlfors' Theorem A since in that theorem the images of θ_{σ_i} $(i = 1, 2)$ contain interior points of the strip $0 < \eta < a$. It follows that the sign of strict inequality holds in Theorem II, and the proof of that theorem is complete.

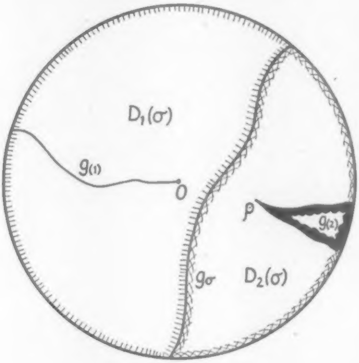

Fig. 4.

9. Proof of Theorem III

The proof of Theorem III begins similarly to that of Theorem I. We use Lemma 1 to show that the curves g_σ of Theorem III, whose total area is finite, have finite length for almost all σ. In this case the closure of g_σ is a simple (or simple closed) Jordan curve, and, since g_σ separates $z = 0$, ρ, it follows that $\bar{g}_\sigma$ forms a cross-cut in $|z| < 1$ dividing $|z| < 1$ into two domains $D_1(\sigma)$, $D_2(\sigma)$ of which the first contains $z = 0$ and the second $z = \rho$. The sets $D_1(\sigma)$, $D_2(\sigma)$ for varying σ form two monotonic sequences of domains, and we proceed as in the proof of Lemma 3 to form the two continua

$$g_{(1)} = \prod_{\sigma \text{ in } E} \overline{D_1(\sigma)}, \qquad g_{(2)} = \prod_{\sigma \text{ in } E} \overline{D_2(\sigma)}$$

which connect $z = 0$, ρ respectively to $|z| = 1$ and contain no point of any g_σ. Since by hypothesis the g_σ separate $z = 0$, ρ in $|z| < 1$, it follows that the boundary points of each g_σ on $|z| = 1$ separate those of $g_{(1)}$, $g_{(2)}$. The complement of $g_{(1)}$, $g_{(2)}$ in $|z| < 1$ is a simply-connected domain, which we denote by D. We call $g_{(1)}$, $g_{(2)}$ the *complementary* continua of D and ρ the *parameter* of D. Let $g^0_{(1)}$, $g^0_{(2)}$ be the segments $-1 < z' < 0, \rho' < z' < 1$ respectively in the z'-plane and let D_0 be the complementary domain in $|z'| < 1$. To prove Theorem III we need

LEMMA 6. *With the notation introduced above, if D with parameter ρ is distinct from D_0, D can be mapped one to one and conformally on a domain D_0 with parameter $\rho' < \rho$, in such a manner that the boundary points of D on $|z| = 1$ correspond to those of D_0 on $|z'| = 1$ and conversely.*

Before proving Lemma 6, I shall show how Theorem III is deduced from the lemma. We can map the domain D_0 on a rectangle $\mathfrak{R}$ $(0 < \mathrm{Rl}\, w < I_0;\ 0 < \mathrm{Im}\, w = 1)$, where $\rho_0(I_0) = \rho'$, and $\rho_0(I)$ is the function defined in § 2, in such a manner that the slits $g^0_{(1)}$, $g^0_{(2)}$ correspond to the sides $\mathrm{Rl}\, w = 0$, I_0 of the rectangle and conversely. By combining this mapping with the mapping of Theorem III and the mapping which maps D on D_0, we obtain a mapping of Ω into the rectangle $\mathfrak{R}$ in such a manner that the transforms of the segments θ_σ have limit points on $\mathrm{Im}\, w = 0$ and on $\mathrm{Im}\, w = 1$. We can therefore apply Lemma 2 and obtain $I_0 \geqslant I$. Hence

$$\rho \geqslant \rho' = \rho_0(I_0) \geqslant \rho_0(I),$$

which proves Theorem III.

10. It remains to prove Lemma 6. The functions which map D_0 on a domain D in the way described in Lemma 6 again form a compact family, so that it is sufficient to show that, if D_1 is any domain D other than D_0, with complementary continua $g^1_{(1)}$, $g^1_{(2)}$ and parameter ρ_1, then there exists another domain D_2, with complementary continua $g^2_{(1)}$, $g^2_{(2)}$ and parameter $\rho_2 < \rho_1$, and which can be mapped one to one and conformally on D_1 so that the boundary points on $g^1_{(1)}$, $g^1_{(2)}$ correspond to those on $g^2_{(1)}$, $g^2_{(2)}$ respectively and conversely.

To prove this we need the idea of the *harmonic measure* $\omega[z, \alpha; D]$ of a boundary arc α of a domain D at an interior point z of D. The expression $\omega[z, \alpha; D]$ may be defined as a harmonic function

in D which tends to 1 as z tends to a boundary point interior to the arc α, and tends to 0 as z tends to an interior point of a different boundary arc. We have

LEMMA 7. *Let D_1, D_2 be two simply-connected domains in the w-plane, let α_1, α_2 be two boundary arcs and w_1, w_2 two interior points of D_1, D_2 respectively such that $\omega[w_1, \alpha_1; D_1] = \omega[w_2, \alpha_2; D_2]$.*

Then D_1 can be mapped one to one and conformally on D_2 so that w_1 maps into w_2 and the boundary arcs α_1, α_2 correspond.

Let Δ be the strip $0 < x < 1$ in the z-plane and let

$$z = z_1(w), \qquad z = z_2(w)$$

map D_1, D_2 on Δ so that the arcs α_1, α_2 map into $x = 0$. Let z_1, z_2 be the images of w_1, w_2 under these transformations. The harmonic measure, being a harmonic function, is invariant under conformal transformations so that we have

$$\omega[w_i, \alpha_i; D_i] = \omega[z_i, x = 0; \Delta] = \mathrm{Rl}\, z_i.$$

Hence we can map Δ on itself by a transformation $z' = z + ic$ which maps the line $x = 0$ into itself and z_1 into z_2. By combining this transformation with the original mappings of D_1, D_2 on Δ, we obtain the desired conformal mapping of D_1 on D_2, and the lemma is proved.

We need also the following (known) result.†

LEMMA 8. *Let $g^2_{(1)}$ be the straight-line segment $-1 < z < 0$ and let $g^1_{(1)}$ be any other acyclic continuum‡ joining $z = 0$ to $|z| = 1$ in $|z| < 1$. Let D_2^*, D_1^* be the domains consisting of those points in $|z| < 1$ not on $g^2_{(1)}$, $g^1_{(1)}$ respectively. Then, if $0 < \rho < 1$, we have*

$$\omega[\rho, g^2_{(1)}; D_2] < \omega[\rho, g^1_{(1)}; D_1].$$

We can now prove Lemma 6. Let $g^1_{(1)}$, $g^1_{(2)}$ be the complementary continua of the domain D of that lemma and suppose, for instance, that $g^1_{(1)}$ is not confined to the negative real axis. Let $g^2_{(1)}$ be the segment $-1 < z < 0$ of the real axis and let D_2^*, D_1^* be the domains consisting of those points of $|z| < 1$ not on $g^2_{(1)}$, $g^1_{(1)}$ respectively. Then Lemma 8 gives

$$\omega[\rho_1, g^2_{(1)}; D_2^*] < \omega[\rho_1, g^1_{(1)}; D_1^*],$$

† R. Nevanlinna, *Eindeutige Analytische Funktionen* (Berlin, 1936), 96. Nevanlinna there gives the solution of a more general problem due to Carleman and Milloux.

‡ i.e. one whose complement is a simply-connected domain.

where ρ_1 is the parameter of D. Also $\omega[\rho, g^2_{(1)}; D^*_2] \to 1$, as $\rho \to 0$. Hence there exists $\rho_2 < \rho_1$ such that

$$\omega[\rho_2, g^2_{(1)}; D^*_2] = \omega[\rho_1, g^1_{(1)}; D^*_1].$$

It then follows from Lemma 7 that we may map D^*_1 one to one and conformally on D^*_2 in such a manner that the points $z = \rho_1, \rho_2$ and the boundary arcs $g^1_{(1)}, g^2_{(1)}$ correspond. In this mapping $g^1_{(2)}$ will map into a continuum $g^2_{(2)}$ joining $z = \rho_2 < \rho_1$ to $|z| = 1$. This completes the proof of Lemma 6 and of Theorem III.

11. Proof of Theorem V

We conclude the paper by an investigation of the numerical properties of the functions $\rho_0(I)$, $\xi_0(I)$, $d_0(I)$, which occur in Theorems I to IV. These can be obtained in terms of the moduli of a system of Jacobian elliptic functions with quarter periods K, iK', where

$$\frac{K}{K'} = I.$$

I shall quote without proof the classical formulae involving these functions which we need. They will all be found in any standard text-book on elliptic functions.† Let

$$v = \operatorname{sn}^2[u \mid K+iK']$$

be the elliptic function corresponding to this lattice. This function maps the rectangle $\mathfrak{R}$ given by $0 < \operatorname{Rl} u < K$, $0 < \operatorname{Im} u < K'$ one to one and conformally on the upper half-plane $\operatorname{Im} v > 0$ in such a manner that the corners $u = 0, K, K+iK', iK'$ on the boundary of the rectangle correspond to $v = 0, 1, 1/k^2, \infty$ on the real v-axis.‡ Here k is the modulus of the system which is real and satisfies $0 < k < 1$ since $\operatorname{sn}^2(u)$ has a real and a purely imaginary quarter-period. If we put

$$\zeta = \frac{1}{\pi}\log\frac{k^2 v}{1-k^2 v} = \xi + i\eta,$$

we obtain a mapping of the rectangle $\mathfrak{R}$ on the strip $0 < \eta < 1$ in such a manner that the points $u = 0, K, K+iK', iK'$ correspond to $\zeta = -\infty, \xi_0, +\infty, i$ where

$$\xi_0 = \frac{1}{\pi}\log\frac{k^2}{1-k^2}.$$

† See for instance P. Appell and E. Lacour, *Théorie des Fonctions Elliptiques* (Paris, 1897). The following references all refer to pages of this book.

‡ Op. cit. 152.

Hence from the definition in § 2, we see that

$$\xi_0(I) = \xi_0 = \frac{1}{\pi}\log\frac{k^2}{1-k^2} = \frac{2}{\pi}\log\frac{k}{k'}, \tag{11.1}$$

where k' is the positive real number defined by $k'^2+k^2 = 1$,† and is the complementary modulus of the lattice.

We proceed to find an expression for $\rho_0(I)$. By means of the transformation

$$t = \left(\frac{1+z}{1-z}\right)^2$$

we map the unit circle $|z| < 1$, cut along the real axis from -1 to 0 and from ρ to 1, on the t-plane, cut along the real axis from $-\infty$ to 1 and from $R = (1+\rho)^2/(1-\rho)^2$ to $+\infty$, so that the boundary points $z = -1, 1$ correspond to $t = 0, \infty$. Then

$$w = \sqrt{\left(\frac{t-1}{R-t}\right)}$$

maps this cut plane on the half-plane $\mathrm{Rl}\, w > 0$ so that the points $t = 0, \infty$ correspond to $w = \mp i(1-\rho)/(1+\rho)$, $\mp i$. This half-plane can be mapped conformally on $\mathrm{Im}\, v > 0$ so that $w = -i(1-\rho)/(1+\rho)$, $-i$, $+i$, $+i(1-\rho)/(1+\rho)$ correspond to $v = 0, 1, 1/k^2, \infty$ provided that

$$\left\{-\frac{1-\rho}{1+\rho}, -1, 1, \frac{1-\rho}{1+\rho}\right\} = \left\{0, 1, \frac{1}{k^2}, \infty\right\},$$

$$\left(\frac{2\rho}{1+\rho}\right)^2 \Big/ 4\left(\frac{1-\rho}{1+\rho}\right) = \frac{\rho^2}{1-\rho^2} = \frac{k^2}{1-k^2},$$

i.e. $\rho = k$. If this condition is satisfied, we can combine the transformations considered above and obtain a one-to-one conformal mapping of the rectangle $\mathfrak{R}$ $(0 < \mathrm{Rl}\, u < K;\ 0 < \mathrm{Im}\, u < K')$ on the unit circle cut from ρ to 1 and from -1 to 0 so that the sides $\mathrm{Rl}\, u = 0$, K of the rectangle correspond to the slits. This is the mapping which was used in § 2 to define the function $\rho_0(I)$ so that we have

$$\rho_0(I) = k, \tag{11.2}$$

$$d_0(I) = \tfrac{1}{2}\log\frac{1+k}{1-k}. \tag{11.3}$$

The formulae (11.1) to (11.3) are the formulae (3.1) to (3.3) of Theorem V.

† Op. cit. 148.

12. It remains to prove (3.4) to (3.9).

The method of proof of these inequalities will be as follows. We shall take a variable, e.g.

$$\frac{1}{I}+\xi_0(I)$$

in the case of (3.4) and show that it varies boundedly and monotonically in the range under consideration, in this case $0 < I \leqslant 1$. It will then follow that, between the values assumed at the end-points, the variable concerned remains in the interior of the range, and this fact will be expressed in the inequalities stated in Theorem V and will show at the same time that these inequalities are 'best possible' in the way described in Theorem V.

We note that an interchange of the quarter-periods K, K' interchanges the moduli k, k' and changes I to $1/I$. Hence we deduce from (11.1)

$$\xi_0(I) = -\xi_0\left(\frac{1}{I}\right) \quad (0 < I < \infty), \tag{12.1}$$

and hence

$$\xi_0(1) = 0. \tag{12.2}$$

To prove (3.4) we put

$$q = e^{-\pi(K'/K)} = e^{-\pi/I} \tag{12.3}$$

and have†

$$\frac{k}{k'} = 4q^{\frac{1}{2}}\left[\frac{(1+q^2)(1+q^4)...}{(1-q)(1-q^3)...}\right]^4.$$

We see in particular that $q^{\frac{1}{2}}k'/k$ decreases steadily from $\frac{1}{4}$ as q increases from 0 to 1. Using (12.2) we see that

$$\frac{1}{I}+\frac{2}{\pi}\log\frac{k}{k'} = \frac{1}{I}+\xi_0(I) = \frac{2}{\pi}\log q^{-\frac{1}{2}}\frac{k}{k'}$$

increases steadily from $(4/\pi)\log 2$ to 1 as I increases from 0 to 1, which proves (3.4). Then (3.5) follows on making use of (12.1).

When $I = 1$, we have $K = K'$, $k = k'$, and, since $k^2+k'^2 = 1$, we deduce

$$k = \rho_0(1) = \frac{1}{\sqrt{2}}. \tag{12.4}$$

We have the expansion†

$$\rho_0(I) = k = 4q^{\frac{1}{2}}\left[\frac{(1+q^2)(1+q^4)...}{(1+q)(1+q^3)...}\right]^4. \tag{12.5}$$

Each of the terms $(1+q^{2n})/(1+q^{2n-1})$ is decreasing for $q < \frac{1}{3}$, so that, by (12.3), $q^{-\frac{1}{2}}\rho_0(I)$ decreases steadily for $0 < I \leqslant 1$. Making use of (12.3), (12.4), and (12.5) we deduce (3.6).

† Op. cit. 127 and 121.

13. To prove the inequalities (3.7) to (3.9), we need the rather more difficult

LEMMA 9. *The function $q(1+k')/(1-k')$ is monotonic increasing in q for $0 < q < \frac{1}{4}$.*

We have†

$$k' = \left(\frac{1-2q+2q^4-\dots}{1+2q+2q^4+\dots}\right)^2 = Z^2 = \left(\frac{X-Y}{X+Y}\right)^2,$$

where $\qquad X = 1+2q^4+\dots, \qquad Y = 2q+2q^9+\dots.$

Thus

$$\frac{d}{dq}\left[q\left(\frac{1+k'}{1-k'}\right)\right] = \frac{d}{dq}\left[q\frac{1+Z^2}{1-Z^2}\right] = \frac{1}{(1-Z^2)^2}[1-Z^4+4qZZ']$$

$$= \frac{8}{(1-Z^2)^2(X+Y)^4}[XY(X^2+Y^2)-q(X^2-Y^2)(XY'-X'Y)]$$

$$> \frac{8X}{(1-Z^2)(X+Y)^4}[Y(X^2+Y^2)-qY'(X^2-Y^2)]$$

$$> \frac{8X}{(1-Z^2)(X+Y)^4}[2q+8q^3-qX^2Y']. \tag{13.1}$$

Now $\qquad X < 1+\dfrac{2q^4}{1-q} < 1+3q^4 \quad (q < \frac{1}{4}).$

Also $\qquad \dfrac{(n+1)^2q^{(n+1)^2}}{n^2q^{n^2}} < 4q^{2n+1} < q \quad (n > 0;\ q < \frac{1}{4}),$

so that

$$qY' < 2q+18q^9[1+q+q^2+\dots] < 2q+24q^9 \quad (q < \tfrac{1}{4}).$$

Hence $\qquad qX^2Y' < (1+6q^4+9q^8)(2q+24q^9) \quad (q < \frac{1}{4}),$

$$< 2q+8q^3 \quad (q < \tfrac{1}{4}).$$

Combining this with (13.1) the lemma follows.

We have $\qquad q\dfrac{1+k'}{1-k'} = q\dfrac{(1+k')^2}{1-k'^2} = q\dfrac{(1+k')^2}{k^2},$

since $k^2+k'^2 = 1$. Hence

$$\lim_{q\to 0} q\frac{1+k'}{1-k'} = \lim_{q\to 0} q\frac{(1+k')^2}{k^2} = 4\lim_{q\to 0}\frac{q}{k^2} = \tfrac{1}{4},$$

by (12.5). Hence using Lemma 9, we have

$$q\frac{1+k'}{1-k'} > \tfrac{1}{4} \quad (0 < q < 1), \tag{13.2}$$

† Op. cit. 125 and 127.

since the inequality is trivial for $q \geqslant \frac{1}{4}$. Since changing I to $1/I$ interchanges k and k', we have from (11.2), (11.3)

$$\rho_0\left(\frac{1}{I}\right) = k', \tag{13.3}$$

$$d_0\left(\frac{1}{I}\right) = \tfrac{1}{2}\log\frac{1+k'}{1-k'}. \tag{13.4}$$

Using (13.2), (13.3) we have

$$1-\rho_0\left(\frac{1}{I}\right) < 4q(1+k') < 8q = 8e^{-\pi/I} \quad (0 < I < \infty),$$

and interchanging I and $1/I$ we deduce

$$\rho_0(I) > 1-8e^{-\pi I} \quad (0 < I < \infty),$$

which is (3.7). Also $k' \to 1$ as $I \to \infty$ so that

$$1-\rho_0(I) \sim 4[1+\rho_0(I)]e^{-\pi I} \sim 8e^{-\pi I}.$$

It remains to prove (3.8) and (3.9). We have from (12.3) and (13.4)

$$d_0\left(\frac{1}{I}\right)-\frac{\pi}{2I} = \tfrac{1}{2}\log\left[q\frac{1+k'}{1-k'}\right].$$

Then (3.8) follows from (13.2) on interchanging I and $1/I$. Also, when $I = 1$, we have $k' = 1/\sqrt{2}$, so that $d_0(1) = \log(1+\sqrt{2})$. Hence using Lemma 9 we have, for $1 \leqslant I < \infty$,

$$d_0(I)-\tfrac{1}{2}\pi I \leqslant \log(1+\sqrt{2})-\tfrac{1}{2}\pi.$$

This proves (3.9) and completes the proof of Theorem V.

Finally, I should like to express my gratitude to the referee for many helpful criticisms and to Mr. H. H. Crann, who kindly drew the diagrams for me.

NOTE ON NON-HOMOGENEOUS QUADRATIC FORMS

By P. VARNAVIDES (*London*)

[Received 2 September 1947]

LET $f(x,y) = ax^2+bxy+cy^2$ be an indefinite binary quadratic form, with real coefficients and discriminant $d = b^2-4ac > 0$. A famous theorem of Minkowski states that, for any real x_0, y_0, there exist x, y with

$$x \equiv x_0 \pmod 1, \qquad y \equiv y_0 \pmod 1, \tag{1}$$

such that

$$|f(x,y)| \leqslant \tfrac{1}{4}\sqrt{d}.$$

A method of improving on this result for any particular form has been given by J. Heinhold,† and another method has been given by H. Davenport.‡ I shall denote by $\phi(f)$ the lower bound of the numbers M such that the inequality

$$|f(x,y)| \leqslant M$$

always has a solution with x, y satisfying (1).

A particularly simple and important case is that in which $f(x,y)$ is the principal form of discriminant $4D$ or D, i.e.

$$f(x,y) = x^2-Dy^2 \quad\text{or}\quad x^2+xy-\tfrac{1}{4}(D-1)y^2, \tag{2}$$

according as $D \equiv 2$ or $3 \pmod 4$ or $D \equiv 1 \pmod 4$ and D is a square-free positive integer. In other words, $f(x,y)$ is the norm of the general integer of the field $k(\sqrt{D})$. In this case, Heinhold's result is: (*a*) If $D \equiv 2$ or $3 \pmod 4$, and we express D, as we can uniquely, in the form $D = q^2+p$, where p, q are positive integers with $p \leqslant 2q$, then

$$\phi(f) \leqslant \tfrac{1}{4}\max(p, 2q+1-p). \tag{3}$$

(*b*) If $D \equiv 1 \pmod 4$, and we express D, as we can uniquely, in the form $D = (2q+1)^2+4p$, where $p \leqslant 2q+1$, then

$$\phi(f) \leqslant \tfrac{1}{4}\max(p, 2q+2-p). \tag{4}$$

Using these results, Heinhold compiles a table§ of upper bounds $\phi_1(f)$ for $\phi(f)$ for all values of D less than 100. Some of the results are exact, in the sense that values of x_0, y_0 can be named such that $|f(x,y)| \geqslant \phi_1(f)$ for all x, y satisfying (1).

† Heinhold, 'Verallgemeinerung und Verschärfung eines Minkowskischen Satzes', *Math. Zeits.* 44 (1939), 659–88.

‡ Davenport, 'Non-homogeneous binary quadratic forms', *Proc. K. Akad. Wet. Amsterdam*, 49 (1946), 815–21.

§ Heinhold, loc. cit. 683–4.

Davenport's result† is as follows. Let f_1 be any value of $|f(x,y)|$ arising from co-prime integral values of x, y which satisfies $0 < f_1 < \surd d$. Let

$$\alpha^2 = \frac{d}{16f_1^2}. \tag{5}$$

Then

$$\phi(f) \leqslant \begin{cases} \tfrac{1}{4}f_1 & \text{if } \alpha^2 \leqslant \tfrac{1}{4}, \\ \alpha^2 f_1 & \text{if } \tfrac{1}{4} \leqslant \alpha^2 \leqslant \tfrac{1}{2}, \\ f_1\sqrt{(\alpha^2-\tfrac{1}{4})} & \text{if } \alpha^2 \geqslant \tfrac{1}{2}. \end{cases} \tag{6}$$

We first observe that the results of Heinhold, stated above, can be deduced from the first clause alone of (6). For in case (a), we take

$$f_1 = |f(q,1)| = |q^2-D| = p,$$

or

$$f_1 = |f(q+1,1)| = (q+1)^2-D = 2q+1-p$$

according as $p > q$ or $p \leqslant q$. Then

$$\alpha^2 = \frac{q^2+p}{4p^2} \quad \text{or} \quad \alpha^2 = \frac{q^2+p}{4(2q+1-p)^2}$$

respectively, and these satisfy $\alpha^2 < \tfrac{1}{4}$. Hence $\phi(f) \leqslant \tfrac{1}{4}f_1$, which gives (3). We deal similarly with case (b), on defining f_1 as above.

The main object of this note is to prove the following improved form of Davenport's result, the improvement being in the third clause of (6). We then apply this to the quadratic forms (2), and obtain improved estimates for $\phi(f)$ for fourteen of the values of D in Heinhold's tables.

THEOREM. *Let f_1 be any value of $|f(x,y)|$ which corresponds to co-prime integral values of x, y and which satisfies $0 < f_1 < \surd d$. Let α^2 be defined by* (5), *and let n be any positive integer. Then*

$$\phi(f) \leqslant \begin{cases} \tfrac{1}{4}f_1 & \text{if } \alpha^2 \leqslant \tfrac{1}{4}, \\ \alpha^2 f_1 & \text{if } \tfrac{1}{4} \leqslant \alpha^2 \leqslant \tfrac{1}{2}, \\ \tfrac{1}{2}nf_1 & \text{if } \tfrac{1}{4}(n^2+1) \leqslant \alpha^2 \leqslant \tfrac{1}{4}(n^2+2n), \\ (\alpha^2-\tfrac{1}{4}n^2)f_1 & \text{if } \tfrac{1}{4}(n^2+2n) \leqslant \alpha^2 \leqslant \tfrac{1}{4}\{(n+1)^2+1\}. \end{cases} \tag{7}$$

The proof depends on the following

LEMMA.‡ *If $A \geqslant \tfrac{1}{4}$ and*

$$\beta^2 \leqslant A+\tfrac{1}{4}[2A]^2, \tag{8}$$

where $[2A]$ denotes the integral part of $2A$, then for any X_0 there exists $X \equiv X_0 \pmod 1$ satisfying

$$|X^2-\beta^2| \leqslant A. \tag{9}$$

† Davenport, loc. cit., Theorem 1.

‡ This is Lemma 5 of Davenport's paper, 'Non-homogeneous ternary quadratic forms', accepted for publication in *Acta Mathematica*.

To prove the theorem, we observe first that we can make a linear transformation from (x,y) to (X,Y), with integral coefficients and determinant ± 1, so that

$$f(x,y) = a_1 X^2 + b_1 XY + c_1 Y^2 \tag{10}$$

where $|a_1| = f_1$. This transformation changes the conditions (1) into similar conditions on X, Y, say

$$X \equiv X_0 \pmod 1 \quad \text{and} \quad Y \equiv Y_0 \pmod 1.$$

We can write (10) in the form

$$f(x,y) = \pm f_1\left((X+\theta Y)^2 - \frac{dY^2}{4f_1^2}\right).$$

We choose $Y \equiv Y_0 \pmod 1$ with $|Y| \leqslant \frac{1}{2}$ and have

$$f(x,y) = \pm f_1\{(X+\theta Y)^2 - \beta^2\},$$

where
$$\beta^2 = \frac{d}{4f_1^2} Y^2 \leqslant \alpha^2,$$

α^2 being defined by (5).

It follows from the lemma that we can choose $X \equiv X_0 \pmod 1$ so that

$$|(X+\theta Y)^2 - \beta^2| \leqslant A(\beta),$$

where $A(\beta)$ is the least real number which satisfies (8) with $A(\beta) \geqslant \frac{1}{4}$. We therefore have

$$\phi(f) \leqslant f_1 \max A(\beta), \tag{11}$$

the maximum being taken for $0 \leqslant \beta \leqslant \alpha$.

Case 1. If $\alpha^2 \leqslant \frac{1}{4}$, (8) is satisfied when $A = \frac{1}{4}$, and so (11) implies the first result stated in (7).

Case 2. If $\frac{1}{4} \leqslant \alpha^2 \leqslant \frac{1}{2}$, (8) is satisfied when $A = \alpha^2$, and so we obtain the second result stated in (7).

Case 3. Suppose that

$$\tfrac{1}{4}(n^2+1) \leqslant \alpha^2 \leqslant \tfrac{1}{4}(n^2+2n)$$

where n is a positive integer. Then (8) is satisfied by $A = \frac{1}{2}n$, since it then reduces to

$$\beta^2 \leqslant \tfrac{1}{2}n + \tfrac{1}{4}n^2,$$

which is true. This gives the third result in (7).

Case 4. Suppose that

$$\tfrac{1}{4}(n^2+2n) \leqslant \alpha^2 \leqslant \tfrac{1}{4}\{(n+1)^2+1\}.$$

Then (8) is satisfied with $A = \alpha^2 - \frac{1}{4}n^2$. For this number is at least $\frac{1}{2}n$ and so satisfies the condition $A \geqslant \frac{1}{4}$; and (8) is

$$\beta^2 \leqslant \alpha^2 - \tfrac{1}{4}n^2 + \tfrac{1}{4}[2\alpha^2 - \tfrac{1}{2}n^2]^2. \tag{12}$$

Now $$2\alpha^2-\tfrac{1}{2}n^2 \geqslant \tfrac{1}{2}(n^2+2n)-\tfrac{1}{2}n^2 = n.$$

Hence (12) is satisfied if

$$\beta^2 \leqslant \alpha^2-\tfrac{1}{4}n^2+\tfrac{1}{4}n^2,$$

which is so. This gives the fourth result in (7), and the theorem is proved.

I now apply the theorem to the forms (2) for those values of D in which it gives a better estimate for $\phi(f)$ than the estimate in Heinhold's table. I indicate the value of f_1 used, and state which case of (7) arises and state Heinhold's estimate in parenthesis.

$D = 11$. We take $f_1 = |f(3,1)| = 2$ and have Case 3, with $n = 1$ giving $\phi(f) \leqslant 1$ $(\frac{5}{4})$.

$D = 46$. We take $f_1 = |f(27,4)| = 7$ and have Case 1, giving $\phi(f) \leqslant \frac{7}{4}$ $(\frac{5}{2})$.

$D = 51$. We take $f_1 = |f(7,1)| = 2$ and have Case 3, with $n = 3$, giving $\phi(f) \leqslant 3$ $(\frac{13}{4})$.

$D = 58$. We take $f_1 = f(15,2) = 7$ and have Case 2, giving, $\phi(f) \leqslant \frac{29}{14}$ $(\frac{9}{4})$.

$D = 67$. We take $f_1 = f(41,\ 5) = 6$ and have Case 2, giving $\phi(f) \leqslant \frac{67}{24}$ $(\frac{7}{2})$.

$D = 83$. We take $f_1 = |f(9,1)| = 2$ and have Case 3 with $n = 4$, giving $\phi(f) \leqslant 4$ $(\frac{17}{4})$.

$D = 86$. We take $f_1 = |f(9,1)| = 5$ and have Case 4 with $n = 1$, giving $\phi(f) \leqslant \frac{61}{20}$ $(\frac{7}{2})$.

$D = 87$. We take $f_1 = |f(9,1)| = 6$ and have Case 3 with $n = 1$, giving $\phi(f) \leqslant 3$ $(\frac{13}{4})$.

$D = 94$. We take $f_1 = f(10,1) = 6$ and have Case 3 with $n = 1$, giving $\phi(f) \leqslant 3$ $(\frac{13}{4})$.

$D = 29$. We take $f_1 = |f(2,1)| = 1$ and have Case 3 with $n = 2$, giving $\phi(f) \leqslant 1$ $(\frac{5}{4})$.

$D = 33$. We take $f_1 = f(5,2) = 3$ and have Case 1, giving $\phi(f) \leqslant \frac{3}{4}$ (1).

$D = 53$. We take $f_1 = |f(3,1)| = 1$ and have Case 3 with $n = 3$, giving $\phi(f) \leqslant \frac{3}{2}$ $(\frac{7}{4})$.

$D = 57$. We take $f_1 = f(10,3) = 4$ and have Case 1, giving $\phi(f) \leqslant 1$ $(\frac{3}{2})$.

$D = 89$. We take $f_1 = f(17,4) = 5$ and have Case 1, giving $\phi(f) \leqslant \frac{5}{4}$ (2).

In conclusion, it should be pointed out that the method of the present paper still does not commonly give the best possible inequality for ϕ. I have investigated the first two cases, namely $\sqrt{7}$ and $\sqrt{11}$, in which the best possible inequality for ϕ was unknown, using a more elaborate method due to Davenport. In this way I have proved that the best possible inequalities for ϕ in these two cases are

$$\phi \leqslant \tfrac{9}{14} \quad \text{for} \quad k(\sqrt{7}) \qquad \text{and} \qquad \phi \leqslant \tfrac{19}{22} \quad \text{for} \quad k(\sqrt{11}).$$

In conclusion I wish to express my gratitude to Professor Davenport for the help he has given me in the preparation of this note.

THE ABEL SUMMATION OF CERTAIN DIRICHLET SERIES

By F. V. ATKINSON (*Oxford*)

[Received 1 October 1947]

1. Introduction

It is well known† that to every Dirichlet series of the form $\sum a_n n^{-s}$ there corresponds an abscissa of convergence σ_0 such that the series converges when $\sigma > \sigma_0$ and diverges when $\sigma < \sigma_0$. By including the cases $\sigma_0 = -\infty$, and $\sigma_0 = +\infty$ we allow for the possibilities that the series may converge respectively everywhere or nowhere. Similar results hold for summability by various means.‡

In this paper I consider a modified form of Abel summation, namely, expressions of the form

$$\lim_{\delta \to 0} \left\{ \sum_{n=1}^{\infty} a_n n^{-s} e^{-n\delta} - f(s, \delta) \right\}.$$

In the case of the Riemann zeta function I show that such an expression can extend the range of the original series to the whole plane, apart from isolated points. I consider also the Dirichlet series associated with the partition function. In this case the original series diverges everywhere, and only the method of summation enables us to show that the function exists and that it has a transformation theory.

2. The zeta function

I prove the following result:

As $\delta \to 0$ through positive values,

$$\lim \left\{ \sum_{n=1}^{\infty} n^{-s} e^{-n\delta} - \Gamma(1-s)\delta^{s-1} \right\} = \zeta(s)$$

except when s is a positive integer.

Let δ be positive and s not a positive integer. Then§

$$1 = -\frac{\Gamma(1-s)}{2\pi i} \int_C (-z)^{s-1} e^{-z}\, dz,$$

where the contour C begins and ends at $+\infty$, and encircles the origin

† See, for instance, Titchmarsh (**1**), Chapter 9.
‡ Hardy and Riesz (**2**).
§ Whittaker and Watson (**3**), 245.

in a positive sense, and where $\arg(-z)$ vanishes on the negative real axis. Hence

$$\sum_{n=1}^{\infty} n^{-s}e^{-n\delta} = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^{z+\delta}-1}\,dz,$$

provided that the contour C passes between the points O and $-\delta$. If we deform the contour C to C' so as to include both the points O and $-\delta$, we get

$$\sum_{n=1}^{\infty} n^{-s}e^{-n\delta} = \Gamma(1-s)\delta^{s-1} - \frac{\Gamma(1-s)}{2\pi i}\int_{C'} \frac{(-z)^{s-1}}{e^{z+\delta}-1}\,dz,$$

and so

$$\lim_{\delta\to 0}\left\{\sum_{n=1}^{\infty} n^{-s}e^{-n\delta} - \Gamma(1-s)\delta^{s-1}\right\} = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^{z}-1}\,dz,$$

the right-hand side being an analytic function of s except possibly when s is a positive integer. But, if the real part of s is greater than 1 and if s is not an integer, we have by similar reasoning the well-known result†

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_C \frac{(-z)^{s-1}}{e^{z}-1}\,dz.$$

The result stated now follows.

3. The Dirichlet series associated with the partition function

As usual we denote by $p(n)$ the number of unrestricted partitions of n into positive integers, irrespective of order. Then it is known‡ that, if $|\arg z| < \frac{1}{2}\pi$, then

$$\sum_{n=0}^{\infty} p(n)e^{-2\pi(n-1/24)z} = \sqrt{z}\sum_{n=0}^{\infty} p(n)e^{-2\pi(n-1/24)/z}. \tag{3.1}$$

In a previous paper§ I deduced from (3.1) a 'summation formula' of the type that several writers|| have shown to be connected with the functional transformations of Dirichlet series. I show now that the corresponding Dirichlet series does in fact exist and that it satisfies a functional equation of the appropriate kind. The series in question is

$$\sum_{n=1}^{\infty} p(n)\left(n-\frac{1}{24}\right)^{-s}.$$

† Whittaker and Watson (**3**), 266.
‡ Cf. Hardy and Ramanujan (**4**), 80.
§ Atkinson (**5**).
|| See, for instance, Ferrar (**6**), Guinand (**7**).

In view of the result†

$$\log p(n) \sim 2\pi\sqrt{n}/\sqrt{6}$$

it is plain that the series is everywhere divergent. The result I prove runs as follows:

Let $\psi(s)$ denote the limit, as $h \to 0$ through positive values, of

$$\Gamma(s)(2\pi)^{-s}\sum_{n=1}^{\infty} p(n)\left(n-\frac{1}{24}\right)^{-s} e^{-2\pi h(n-1/24)} - f(s,h),$$

where

$$f(s,h) = -\tfrac{1}{2}i\sec\pi s\int_0^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz \quad (|\arg s| < \tfrac{1}{2}\pi) \tag{3.2}$$

$$= -\tfrac{1}{2}i\sec\pi s\Gamma(s)e^{\pi/24h}h^{s+\frac{1}{2}}\left(\frac{\pi}{12}\right)^{\frac{1}{4}} \times$$

$$\times\left\{e^{-\frac{1}{4}\pi i}W_{-s-\frac{1}{4},-\frac{3}{4}}\left(\frac{\pi}{12h}e^{\pi i}\right) - e^{\frac{1}{4}\pi i}W_{-s-\frac{1}{4},-\frac{3}{4}}\left(\frac{\pi}{12h}e^{-\pi i}\right)\right\}. \tag{3.3}$$

Then $\psi(s)$ is an analytic function of s, except for simple poles at the points $0, -1, -2,...,$ *and* $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2},....$ *Furthermore,*

$$\psi(s) = \psi(-\tfrac{1}{2}-s).$$

Here the W-function is Whittaker's confluent hypergeometric function. Further explanations of the notation are given below.

As noted in my previous paper, in the case of the 'summation formula', (3.1) is a particular case of a more general formula, and the result on Dirichlet series admits of a corresponding generalization.

4. Proof of the results of § 3

Assume to begin with that $|\arg s| < \frac{1}{2}\pi$, and that $2s$ is not an integer. Let h be real and positive. Then

$$\Gamma(s)(2\pi)^{-s}\sum_{n=1}^{\infty} p(n)\left(n-\frac{1}{24}\right)^{-s} e^{-2\pi h(n-1/24)}$$
$$= \int_0^{\infty} z^{s-1}\sum_{n=1}^{\infty} p(n)e^{-2\pi(n-1/24)(z+h)}\,dz.$$

Plainly, as $h \to 0$,

$$\int_1^{\infty} z^{s-1}\sum_{n=1}^{\infty} p(n)e^{-2\pi(n-1/24)(z+h)}\,dz \to \int_1^{\infty} z^{s-1}\sum_{n=1}^{\infty} p(n)e^{-2\pi z(n-1/24)}\,dz.$$

† Hardy and Ramanujan (4), 89.

The integral over $(0,1)$ may be written

$$\int_0^1 z^{s-1} \sum_{n=0}^{\infty} p(n) e^{-2\pi(n-1/24)(h+z)}\, dz - \int_0^1 z^{s-1} e^{\pi(h+z)/12}\, dz$$

$$= \int_0^1 z^{s-1}\sqrt{(h+z)} \sum_{n=0}^{\infty} p(n) e^{-2\pi(n-1/24)/(h+z)}\, dz - \int_0^1 z^{s-1} e^{\pi(h+z)/12}\, dz$$

$$= \int_0^1 z^{s-1}\sqrt{(h+z)} \sum_{n=1}^{\infty} p(n) e^{-2\pi(n-1/24)/(h+z)}\, dz +$$

$$+ \int_0^1 z^{s-1}\sqrt{(h+z)} e^{\pi/12(h+z)}\, dz - \int_0^1 z^{s-1} e^{\pi(h+z)/12}\, dz. \tag{4.1}$$

We may now make $h \to 0$ in the first integral in (4.1), getting

$$\int_0^1 z^{s-1}\sqrt{(h+z)} \sum_{n=1}^{\infty} p(n) e^{-2\pi(n-1/24)/(h+z)}\, dz \to \int_0^1 z^{s-\frac{1}{2}} \sum_{n=1}^{\infty} p(n) e^{-2\pi(n-1/24)/z}\, dz$$

$$= \int_1^{\infty} z^{-s-\frac{3}{2}} \sum_{n=1}^{\infty} p(n) e^{-2\pi z(n-1/24)}\, dz,$$

the series and the integral being uniformly convergent.

We transform the remaining integrals in (4.1) into loop integrals. By $\int_a^{(b+)}$ we understand that the contour begins and ends at a and encircles b in a positive sense; we use a corresponding notation for negative circuits. We get

$$\int_0^1 z^{s-1} e^{\pi(h+z)/12}\, dz = -\frac{1}{2i\sin \pi s} \int_1^{(0+)} (-z)^{s-1} e^{\pi(h+z)/12}\, dz,$$

$$\int_0^1 z^{s-1}\sqrt{(h+z)} e^{\pi/12(h+z)}\, dz$$

$$= -\frac{1}{2i\sin \pi(s+\frac{1}{2})} \left\{ \int_1^{(-h+)} - \int_0^{(-h+)} \right\} (-z)^{s-1}(-h-z)^{\frac{1}{2}} e^{\pi/12(h+z)}\, dz,$$

where $\arg(-z)$, $\arg(-h-z)$ are both zero when z is real and less than $-h$. But, as $h \to 0$,

$$\int_1^{(0+)} (-z)^{s-1} e^{\pi(h+z)/12}\, dz \to \int_1^{(0+)} (-z)^{s-1} e^{\pi z/12}\, dz,$$

$$\int_1^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz \to \int_1^{(0+)} (-z)^{s-\frac{1}{2}}e^{\pi/12z}\,dz$$

$$= \int_1^{(0+)} (-z)^{-s-\frac{1}{2}}e^{\pi z/12}\,dz.$$

Collecting these results and rearranging, we get

$$\lim_{h\to 0}\left\{\Gamma(s)(2\pi)^{-s}\sum_{n=1}^{\infty} p(n)\left(n-\frac{1}{24}\right)^{-s} e^{-2\pi h(n-1/24)} - \right.$$
$$\left. -\frac{1}{2i\cos\pi s}\int_0^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz\right\}$$
$$= \int_1^{\infty} z^{s-1}\sum_{n=1}^{\infty} p(n)e^{-2\pi z(n-1/24)}\,dz + \int_1^{\infty} z^{-s-\frac{1}{2}}\sum_{n=1}^{\infty} p(n)e^{-2\pi z(n-1/24)}\,dz +$$
$$+\frac{1}{2i\sin\pi s}\int_1^{(0+)} (-z)^{s-1}e^{\pi z/12}\,dz + \frac{1}{2i\sin\pi(-s-\frac{1}{2})}\int_1^{(0+)} (-z)^{-s-\frac{1}{2}}e^{\pi z/12}\,dz. \tag{4.2}$$

The expression on the left-hand side of (4.2) is $\psi(s)$ as defined in § 3, while that on the right-hand side is an analytic function of s except for simple poles at the points $0, -1, -2,\ldots$ and $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2},\ldots$, and is unaffected by replacing s by $-s-\frac{1}{2}$. Hence $\psi(s)$ satisfies the functional equation and is analytic except for simple poles at the above-named points.

It remains to verify that the expressions (3.2) and (3.3) for $f(s,h)$ are equivalent. In (3.2) we change the variable, putting

$$y = h^{-1}-(h+z)^{-1}, \qquad z = h^2y(1-hy)^{-1}, \qquad dz = h^2(1-hy)^{-2}\,dy.$$

Then

$$\int_0^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz$$
$$= \int_0^{(1/h-)} \left(-\frac{h^2y}{1-hy}\right)^{s-1}\left(-\frac{h}{1-hy}\right)^{\frac{1}{2}} e^{\pi(1/h-y)/12}\frac{h^2\,dy}{(1-hy)^2}$$
$$= h^{2s+\frac{1}{2}}e^{\pi/12h}\int_0^{(1/h-)} y^{s-1}(hy-1)^{-s-\frac{1}{2}}e^{-\pi y/12}\,dy,$$

where $\arg y = 0$, $\arg(hy-1) = 0$, for real $y > h^{-1}$.

Write $$I_1, I_2 = \int\limits_0^{\infty} y^{s-1}(1-hy)^{-s-\frac{3}{2}}e^{-\pi y/12}\,dy,$$

wherein the paths of integration pass initially along the positive real axis, encircle the point h^{-1} in a negative and a positive sense respectively, and then proceed along the real axis to $+\infty$. The value of the integrand is fixed in each case by the requirements $\arg y = 0$, $\arg(1-hy) = 0$, for $0 < y < h^{-1}$. We have then

$$\int\limits_0^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz = h^{2s+\frac{1}{2}}e^{\pi/12h}\{e^{-\pi i(s+\frac{3}{2})}I_1-e^{\pi i(s+\frac{3}{2})}I_2\}.$$

But, on using a standard expression† for the Whittaker W-functions,

$$I_1 = \left(\frac{\pi}{12}\right)^{-s}\int\limits_0^{\infty+i} y^{s-1}\left\{1+y\Big/\left(-\frac{\pi}{12h}\right)\right\}^{-s-\frac{3}{2}}e^{-y}\,dy$$

$$= \left(\frac{\pi}{12}\right)^{-s}\Gamma(s)e^{-\pi/24h}\left(\frac{\pi}{12h}e^{\pi i}\right)^{s+\frac{1}{4}}W_{-s-\frac{1}{4},-\frac{3}{4}}\left(\frac{\pi}{12h}e^{\pi i}\right),$$

where by the expression $W_{k,m}(ze^{\pi i})$, z being real, is meant the limit of $W_{k,m}(ze^{i\theta})$ as $\theta \to \pi$ from below. We have similarly a conjugate expression for I_2. Hence

$$\int\limits_0^{(-h+)} (-z)^{s-1}(-h-z)^{\frac{1}{2}}e^{\pi/12(h+z)}\,dz = \Gamma(s)e^{\pi/24h}\left(\frac{\pi}{12}\right)^{\frac{1}{4}}h^{s+\frac{1}{4}}\times$$

$$\times\left\{e^{-\frac{3}{4}\pi i}W_{-s-\frac{1}{4},-\frac{3}{4}}\left(\frac{\pi}{12h}e^{\pi i}\right)-e^{\frac{3}{4}\pi i}W_{-s-\frac{1}{4},-\frac{3}{4}}\left(\frac{\pi}{12h}e^{-\pi i}\right)\right\}$$

which is the result required for the equivalence of (3.2) and (3.3). This completes the proof.

† Whittaker and Watson (**3**), 340.

REFERENCES

1. E. C. Titchmarsh, *Theory of Functions* (Oxford, 1932).
2. G. H. Hardy and M. Riesz, *General Theory of Dirichlet Series* (Cambridge, 1915), Cambridge Tracts, no. 18.
3. E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927, 4th ed.).
4. G. H. Hardy and S. Ramanujan, *Proc. London Math. Soc.* (2), 17 (1918), 75–115.
5. F. V. Atkinson, *J. of London Math. Soc.* 14 (1939), 175–84.
6. W. L. Ferrar, *Compositio Math.* 4 (1937), 394–405.
7. A. P. Guinand, *Proc. London Math. Soc.* (2), 43 (1937), 439–48.